The Interface Control Domain Decomposition (ICDD) method for the Stokes problem

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1. INTRODUCTION

2. PROBLEM SETTING

Let \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)) be an open bounded domain with Lipschitz boundary \( \partial \Omega \). We assume that \( \overline{\Omega} = \overline{\Gamma_D} \cup \overline{\Gamma_N} \) with \( \Gamma_D \cap \Gamma_N = \emptyset \) and that \( \Gamma_D \neq \emptyset \) while \( \Gamma_N \) might be empty. We consider the Stokes problem:

\[
\begin{align*}
\begin{cases}
- \text{div } \mathbf{T}(\mathbf{u}, p) & = \mathbf{f} \quad \text{in } \Omega, \\
\text{div } \mathbf{u} & = 0 \quad \text{in } \Omega, \\
\mathbf{u} & = \phi_D \quad \text{on } \Gamma_D, \\
\mathbf{T}(\mathbf{u}, p) \cdot \mathbf{n} & = \phi_N \quad \text{on } \Gamma_N,
\end{cases}
\end{align*}
\]

(1)

where \( \mathbf{T}(\mathbf{u}, p) = 2\nu \nabla \mathbf{u} - \rho \mathbf{I} \) is the Cauchy stress tensor being \( \nabla \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \), \( \nu > 0 \) is the fluid viscosity, \( \mathbf{u} \) its velocity and \( p \) its pressure and \( \mathbf{n} \) is the unit normal vector to \( \partial \Omega \) directed outwards the domain \( \Omega \). We assume that \( \mathbf{f} \in [L^2(\Omega)]^d \), \( \phi_D \in [H^{1/2}(\Gamma_D)]^d \) and \( \phi_N \in [H^{-1/2}(\Gamma_N)]^d \) are assigned functions. If \( \partial \Omega = \Gamma_D \) (i.e., \( \Gamma_N = \emptyset \)), the compatibility condition \( \int_{\Gamma_D} \phi_D \cdot \mathbf{n} = 0 \) must hold, and a further condition on \( p \), e.g.,

\[
\int_{\Gamma_D} p = 0
\]

must be enforced to guarantee the well-posedness of problem (1).

The weak form of problem (1) is: find \( \mathbf{u} \in [H^1(\Omega)]^d \), \( \mathbf{u} = \phi_D \) on \( \Gamma_D \), and \( p \in L^2(\Omega) \) such that, for all \( \mathbf{v} \in [H^1(\Omega)]^d \), \( q \in L^2(\Omega) \),

\[
\begin{align*}
a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) & = \int_{\Gamma_D} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \phi_N \cdot \mathbf{v}, \\
b(q, \mathbf{u}) & = 0,
\end{align*}
\]

(2)

and

\[
b(q, \mathbf{u}) = - \int_{\Gamma_D} q \text{div } \mathbf{v}.
\]

(3)

For simplicity of exposition, in the rest of the paper we will often use the strong form of the Stokes problem, but it must be understood that the analysis is carried out in the weak setting.

We consider an overlapping decomposition of the domain \( \Omega \) in two subdomains \( \Omega_1 \) and \( \Omega_2 \): \( \Omega = \Omega_1 \cup \Omega_2 \). We denote the overlapping region by \( \Omega_{12} = \Omega_1 \cap \Omega_2 \) and let \( \Gamma_i = \partial \Omega_i \setminus \partial \Omega \). Moreover, let \( \Gamma_D^i = \Gamma_D \cap \partial \Omega_i \) and \( \Gamma_N^i = \Gamma_N \cap \partial \Omega_i \) (see figure 1).

![FIG. 1: Representation of the computational domain Ω and of its overlapping splitting.](image)

We reformulate the Stokes problem (1) on the split domain in the following possible ways.

Problem \( P_{\Gamma, i} \):

\[
\begin{align*}
- \text{div } \mathbf{T}(\mathbf{u}_i, p_i) & = \mathbf{f} \quad \text{in } \Omega_i, \; i = 1, 2, \\
\text{div } \mathbf{u}_i & = 0 \quad \text{in } \Omega_i, \; i = 1, 2, \\
\mathbf{u}_i & = \phi_D \quad \text{on } \Gamma_D^i, \; i = 1, 2, \\
\mathbf{T}(\mathbf{u}_i, p_i) \cdot \mathbf{n} & = \phi_N \quad \text{on } \Gamma_N^i, \; i = 1, 2, \\
\mathbf{u}_1 & = \mathbf{u}_2 \quad \text{on } \Gamma_1 \cup \Gamma_2.
\end{align*}
\]

(4)

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In case $\Gamma_i' = \emptyset$ for some $i$, we would supplement $\mathbf{4}$ with the condition
\[
\int_{\Omega_i} p_i = 0
\]
to ensure the well-posedness of the corresponding local problem.

**Problem $\mathcal{P}_{\Gamma, 1}$:**
\[
-\text{div} \, T(u_i, p_i) = f \quad \text{in} \, \Omega_i, \quad i = 1, 2, \\
\text{div} \, u_i = 0 \quad \text{in} \, \Omega_i, \quad i = 1, 2, \\
\begin{align*}
\quad & u_i = \phi_D \quad \text{on} \, \Gamma_D', \quad i = 1, 2, \\
\quad & T(u_i, p_i) \cdot n = \phi_N \quad \text{on} \, \Gamma_N, \quad i = 1, 2, \\
\quad & T(u_1, p_1) \cdot n = T(u_2, p_2) \cdot n \quad \text{on} \, \Gamma_1 \cup \Gamma_2.
\end{align*}
\]
(5)

Condition $\mathbf{5}$ on $\Gamma_1$ should be understood as follows. The normal vector $n$ on $\Gamma_1$ is directed outward of $\Omega_1$ and the normal component of the tensor $T(u_2, p_2)$ is computed upon restricting it to $\Omega_1$. On the other hand, on $\Gamma_2$ the normal vector $n$ is directed outward of $\Omega_2$ and the normal component of the tensor $T(u_1, p_1)$ is taken upon restricting it to $\Omega_1$.

Moreover, we consider the problem:

**Problem $\mathcal{P}_{\Gamma, 1}$:**
\[
-\text{div} \, T(u_i, p_i) = f \quad \text{in} \, \Omega_i, \quad i = 1, 2, \\
\text{div} \, u_i = 0 \quad \text{in} \, \Omega_i, \quad i = 1, 2, \\
\begin{align*}
\quad & u_i = \phi_D \quad \text{on} \, \Gamma_D', \quad i = 1, 2, \\
\quad & T(u_i, p_i) \cdot n = \phi_N \quad \text{on} \, \Gamma_N, \quad i = 1, 2, \\
\quad & u_1 = u_2, \quad u_2 = u_1 \quad \text{on} \, \Gamma_1, \\
\quad & T(u_1, p_1) \cdot n = T(u_2, p_2) \cdot n \quad \text{on} \, \Gamma_2.
\end{align*}
\]
(6)

If $\Gamma_N' = \emptyset$, we should impose
\[
\int_{\Omega_1} p_1 = 0
\]
to guarantee the well-posedness of the Stokes problem in $\Omega_1$.

Let us introduce the following spaces

$V = [H^1(\Omega)]^d$, \quad $V_i = [H^1(\Omega_i)]^d$, \quad $i = 1, 2$

$Q = L^2(\Omega)$, \quad $Q_0 = \{ q \in Q : \int_{\Omega} q = 0 \}$

$Q_i = L^2(\Omega_i)$, \quad $Q_{i, 0} = \{ q \in Q_i : \int_{\Omega_i} q = 0 \}$ \quad $i = 1, 2$

(7)

and the following manifolds

$V_{\phi_D} = \{ \mathbf{v} \in [H^1(\Omega)]^d : \mathbf{v} = \phi_D \text{ on } \Gamma_D \}$

$V_{i, \phi_D} = \{ \mathbf{v} \in [H^1(\Omega_i)]^d : \mathbf{v} = \phi_D \text{ on } \Gamma_D' \}, \quad i = 1, 2,$

(8)

Finally, we set

$V_{i, 0} = \{ \mathbf{v} \in [H^1(\Omega_i)]^d : \mathbf{v} = 0 \text{ on } \Gamma_D' \}, \quad i = 1, 2.$

(9)

To prove that the Stokes problem $\mathbf{1}$ is equivalent to either $\mathbf{4}$, or $\mathbf{5}$, or $\mathbf{6}$, we will denote $\mathbf{w} = u_{1, \Omega_1} - u_{2, \Omega_1}$ and $q = p_{1, \Omega_1} - p_{2, \Omega_1}$ the difference in $\Omega_1$ between the local solutions that satisfies the Stokes equations:

\[
-\text{div} \, T(w, q) = 0 \quad \text{in} \, \Omega_1 \\
\text{div} \, w = 0 \quad \text{in} \, \Omega_1.
\]
(10)

The boundary conditions fulfilled by $w$ and $q$ as well as the spaces to which these functions belong will be specified case by case.

**Assumption 2.1** We suppose that one of the following assumptions is verified: $\Gamma_N = \emptyset$; $\Gamma_N \neq \emptyset$ and $\Gamma_N \cap \partial \Omega_1 \neq \emptyset$; $\Gamma_N \cap \partial \Omega_1 = \emptyset$ with $\Gamma_N \neq \emptyset$ connected.

**Proposition 2.1 (Equivalence between $\mathcal{P}_{\Omega}$ and $\mathcal{P}_{\Gamma, 1}$)**

The Stokes problems $\mathcal{P}_{\Omega}$ and $\mathcal{P}_{\Gamma, 1}$ are equivalent if the boundary conditions satisfy Assumption 2.1. Equivalence holds in the sense that if $(\mathbf{u}, p)$ and $(\mathbf{u}_i, p_i)$ ($i = 1, 2$) are the unique solutions of $\mathcal{P}_{\Omega}$ and $\mathcal{P}_{\Gamma, 1}$, respectively, there exist two uniquely determined constants $C_1, C_2 \in \mathbb{R}$, possibly null, such that, for $i = 1, 2$, $\mathbf{u}_i |_{\Omega_1} = \mathbf{u}_i$ and $p_i |_{\Omega_1} = p_i + C_i$.

**Proof.** We distinguish the different cases.

1. Assume first that $\Gamma_N \cap \partial \Omega_1 \neq \emptyset$. Then, problem $\mathbf{1}$ is well-posed in $(\mathbf{u}, p) \in V_{\phi_D} \times Q$ and the restrictions of its solution to $\Omega_1$, satisfy $\mathbf{4}$ by construction.

Viceversa, for $i = 1, 2$, let $(\mathbf{u}_i, p_i) \in V_{i, \phi_D} \times Q_i$ ($i = 1, 2$) be the solutions of the well-posed local problems

\[
-\text{div} \, T(u_i, p_i) = f \quad \text{in} \, \Omega_i \\
\text{div} \, u_i = 0 \quad \text{in} \, \Omega_i \\
\begin{align*}
\quad & u_i = \phi_D \quad \text{on} \, \Gamma_D, \\
\quad & T(u_i, p_i) \cdot n = \phi_N \quad \text{on} \, \Gamma_N, \\
\quad & u_i = u_j \quad \text{on} \, \Gamma_{ij}, \quad j = 3 - i.
\end{align*}
\]

(10)

By construction, the functions $w$ and $q$ satisfy problem $\mathbf{10}$ with boundary conditions

\[
T(w, q) \cdot n = 0 \quad \text{on} \, \partial \Omega_1 \cap \Gamma_N \\
w = 0 \quad \text{on} \, \partial \Omega_1 \cap \Gamma_N.
\]

This problem is well-posed and admits the unique solution $w = 0$ and $q = 0$, hence $u_1 = u_2$ and $p_1 = p_2$ in $\Omega_1$. Thus, we can set

\[
\mathbf{u} = \begin{cases} 
\mathbf{u}_1 & \text{in} \, \Omega_1 \setminus \Omega_{12} \\
\mathbf{u}_1 & \text{in} \, \Omega_{12} \setminus \Omega_1 \\
\mathbf{u}_2 & \text{in} \, \Omega_2 \setminus \Omega_{12} \\
\mathbf{u}_2 & \text{in} \, \Omega_{12} \setminus \Omega_2
\end{cases}
\]
(11)

and

\[
\mathbf{p} = \begin{cases} 
p_1 & \text{in} \, \Omega_1 \setminus \Omega_{12} \\
p_1 & \text{in} \, \Omega_{12} \setminus \Omega_1 \\
p_2 & \text{in} \, \Omega_2 \setminus \Omega_{12} \\
p_2 & \text{in} \, \Omega_{12} \setminus \Omega_2
\end{cases}
\]
(12)

By construction, functions $\mathbf{u}$ and $\mathbf{p}$ belong to $V_{\phi_D} \times Q$ and they satisfy problem $\mathbf{1}$. Notice that in this case $C_1 = C_2 = 0$. 

Let now \( \Gamma_N \cap \partial \Omega_{12} = \emptyset \) and assume that \( \Gamma_N \) is connected. In this case, either \( \Gamma_N' = \emptyset \) or \( \Gamma_N'' = \emptyset \). We consider the latter case; the former can be treated analogously.

If \((u, p) \in V_{\phi_d} \times Q\) is the solution of \( \mathcal{P}_\Omega \), if we set \( u_i = u|_{\Omega_i}, (i = 1, 2), p_1 = p|_{\Omega_1}, \)

\[
p_2 = p|_{\Omega_2} - \frac{1}{|\Omega_2|} \int_{\Omega_2} p|_{\Omega_2},
\]

we can immediately verify that \((u_i, p_i) \in V_{\phi_d} \times Q_i, (i = 1, 2)\) are solutions of \( \mathcal{P}_{\Gamma_i} \) with \( \int_{\Omega_2} p_2 = 0 \).

Thus, \( C_1 = 0 \) and \( C_2 = -\frac{1}{|\Omega_2|} \int_{\Omega_2} p|_{\Omega_2} \).

Vice versa, let \((u_1, p_1) \in V_{\phi_d} \times Q_1, (u_2, p_2) \in V_{\phi_d} \times Q_2\) be the solutions of \( \mathcal{P}_{\Gamma_1} \). The functions \((w, q)\) satisfy (10) with \( w = 0 \) on \( \partial \Omega_{12} \). Then, \( w = 0 \) and \( q = \text{const} \) in \( \Omega_{12} \). The function \( q \) is uniquely determined by \( \int_{\Omega_1} q = \int_{\Omega_{12}} (p_1 - p_2) \) which implies

\[
q = \frac{1}{|\Omega_{12}|} \int_{\Omega_{12}} (p_1 - p_2).
\]

If we take \( u \) as in (11) and

\[
p = \begin{cases} 
  p_1 & \text{in } \Omega_1 \setminus \Omega_{12} \\
  p_1 + p_2 + q & \text{in } \Omega_{12} \\
  p_2 + q & \text{in } \Omega_2 \setminus \Omega_{12},
\end{cases}
\]

then \((u, p)\) satisfy \( \mathcal{P}_\Omega \) and the thesis follows with \( C_1 = 0 \) and \( C_2 = q \).

Let \((u, p) \in V_{\phi_d} \times Q_0\) be the solution of \( \mathcal{P}_\Omega \). Then, for \( i = 1, 2 \), the functions

\[
u_i = u|_{\Omega_i}, \quad p_i = p|_{\Omega_i} - \frac{1}{|\Omega_i|} \int_{\Omega_i} p|_{\Omega_i},
\]

belong to \( V_{\phi_d} \times Q_i, 0 \) and they satisfy \( \mathcal{P}_{\Gamma_i} \). Thus, \( C_1 = 0 \) and \( C_2 = -\frac{1}{|\Omega_2|} \int_{\Omega_2} p|_{\Omega_2} \).

Vice versa, let \((u_i, p_i) \in V_{\phi_d} \times Q_i, 0 \) be solutions of \( \mathcal{P}_{\Gamma_i} \). Then, the functions \( w \) and \( q \) satisfy (10) with boundary condition \( w = 0 \) on \( \partial \Omega_{12} \). Then, \( w = 0 \) in \( \Omega_{12} \) and \( q = \text{const} \) in \( \Omega_{12} \). The constant \( q \) is uniquely determined by

\[
\int_{\Omega_{12}} q = \int_{\Omega_{12}} (p_1 - p_2)
\]

that is

\[
q = \frac{1}{|\Omega_{12}|} \int_{\Omega_{12}} (p_1 - p_2).
\]

If we define the constants

\[
C_1 = \frac{1}{|\Omega|} \left( \int_{\Omega_{12}} p_2 - |\Omega \setminus \Omega_1| q \right)
\]

and

\[
C_2 = \frac{1}{|\Omega|} \left( \int_{\Omega_{12}} p_2 + |\Omega_1| q \right),
\]

since \( C_2 - C_1 = q \), then \( p_1 + C_1 = p_2 + C_2 \) in \( \Omega_{12} \).

Thus, we can easily verify that the functions \( u \) and \( p \) defined respectively as in (11) and as

\[
p = \begin{cases} 
  p_1 + C_1 & \text{in } \Omega_1 \setminus \Omega_{12} \\
  p_1 + C_1 + p_2 + C_2 & \text{in } \Omega_{12} \\
  p_2 + C_2 & \text{in } \Omega_2 \setminus \Omega_{12}
\end{cases}
\]

are solutions of \( \mathcal{P}_{\Omega} \) with \( \int_{\Omega} p = 0 \).

\[ \square \]

Remark 2.1 If \( \partial \Omega_{12} \cap \Gamma_N = \emptyset \) and \( \Gamma_N' \neq \emptyset \) \((i = 1, 2)\), problems \( \mathcal{P}_\Omega \) and \( \mathcal{P}_{\Gamma_i} \) are not equivalent.

In fact, if \((u_i, p_i)\) are the solutions of \( \mathcal{P}_{\Gamma_i} \), the functions \( w \) and \( q \) satisfy (10) with boundary condition \( w = 0 \) on \( \partial \Omega_{12} \). Then, \( w = 0 \) and \( q = \text{const} \) in \( \Omega_{12} \) with \( q \) uniquely given by

\[
q = \frac{1}{|\Omega_{12}|} \int_{\Omega_{12}} (p_1 - p_2).
\]

Then, proceeding similarly to the third case of the proof of Proposition 2.1 there exist two unique constants \( C_1, C_2 \) with \( q = C_2 - C_1 \) so that we can define \( u \) and \( p \) as in (11) and (12), respectively. The Neumann boundary conditions in \( \mathcal{P}_{\Gamma_1} \) imply

\[
T(u_i, p_i) \cdot n = \phi_N \quad \text{on } \Gamma_N^i
\]

and, by definition of \( u \) and \( p \), we have

\[
T(u, p) \cdot n = \phi_N + C_i n \quad \text{on } \Gamma_N.
\]

Thus, \((u, p)\) satisfy problem \( \mathcal{P}_\Omega \) if and only if \( C_1 = C_2 = 0 \), but we cannot guarantee that this condition is fulfilled.

Proposition 2.2 (Equivalence between \( \mathcal{P}_\Omega \) and \( \mathcal{P}_{\Gamma_i, j} \)) If \( \partial \Omega_1 \cap \partial \Omega_2 \neq \emptyset \), the Stokes problems \( \mathcal{P}_\Omega \) and \( \mathcal{P}_{\Gamma_{i, j}} \) are equivalent in the sense that there exist unique constants \( C_1, C_2 \in \mathbb{R} \) such that \( u|_{\Omega_i} = u \) and \( p|_{\Omega_i} = p_i + C_i (u, p) \) and \((u_i, p_i) \) \((i = 1, 2)\) being, respectively, the unique solutions of \( \mathcal{P}_\Omega \) and \( \mathcal{P}_{\Gamma_{i, j}} \) with boundary conditions

\[
w = 0 \quad \text{on } \partial \Omega_{12} \cap \partial \Omega
\]

\[
T(w, q) \cdot n = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2.
\]
This problem is well-posed and its solution is $w = 0$ and $q = 0$. Thus, $u_1 = u_2$ and $p_1 = p_2$ in $\Omega_{12}$ and we can define velocity $u$ and a pressure $p$ analogously to (1) and (2). However, the function $\tilde{p}$ would belong to $Q$ but not to $Q_0$, so that we define

$$C_1 = C_2 = -\frac{1}{|\Omega|} \int_{\Omega} \tilde{p}$$

and $p = \tilde{p} + C_1$ to recover the null average.

**Remark 2.2** Problems $P_T$ and $P_{T,f}$ are not equivalent if $\partial \Omega_{12} \cap \Gamma_D = \emptyset$. In fact, in this case problem (10) in $\Omega_{12}$ would be supplemented with the boundary condition $T(w, q) \cdot n = 0$ on $\partial \Omega_{12}$ which has infinite non-trivial solutions that may differ one from another not only by a constant.

**Proposition 2.3 (Equivalence between $P_\Omega$ and $P_{T,f}$)**

The Stokes problems $P_\Omega$ and $P_{T,f}$ are equivalent if either $\Gamma_N = \emptyset$, or $\Gamma_N \cap \partial \Omega_{12} \neq \emptyset$, or $\Gamma_N \cap \partial \Omega_{12} = \emptyset$ and $\Gamma' \neq \emptyset$. Equivalence holds in the sense that if $(u, p)$ and $(u, p)$ $(i = 1, 2)$ are the unique solutions of $P_\Omega$ and $P_{T,f}$, respectively, then there exist two uniquely determined constants $C_1, C_2 \in \mathbb{R}$ possibly null, such that, for $i = 1, 2$, $u|_{\Omega_i} = u_i$ and $p|_{\Omega_i} = p_i + C_i$.

**Proof.** The proof follows similar steps to those of the previous propositions. Let us only point out that equivalence holds with $C_1 = C_2 = 0$ if $\Gamma_N \neq \emptyset$. Otherwise, if $\Gamma_N = \emptyset$, if $(u, p) \in V_{\phi, N} \times Q_0$ is the solution of $P_\Omega$, then $u_i = u_i|_{\Omega_i}$, $p_2 = p_i|_{\Omega_2}$ and

$$p_1 = p|_{\Omega_1} - \frac{1}{|\Omega_1|} \int_{\Omega_1} p|_{\Omega_1}$$

are the solutions of $P_{T,f}$. Conversely, if $(u_1, p_1) \in V_{1, \phi, N} \times Q_{1,0}$ and $(u_2, p_2) \in V_{2, \phi, N} \times Q_2$ are the solutions of $P_{T,f}$, then we need to set

$$C_1 = C_2 = -\frac{1}{|\Omega_i|} \int_{\Omega_i} p_i|_{\Omega_i}$$

**Remark 2.3** Problems $P_T$ and $P_{T,f}$ are not equivalent if $\partial \Omega_{12} \cap \Gamma_N = \emptyset$, $\Gamma_N = \emptyset$, and $\Gamma' \neq \emptyset$. In fact, if $(u_1, p_1) \in V_{1, \phi, N} \times Q_{1,0}$ and $(u_2, p_2) \in V_{2, \phi, N} \times Q_2$ are the solutions of $P_{T,f}$, then $(w, q)$ satisfy problem (10) in $\Omega_{12}$ with boundary condition $T(w, q) \cdot n = 0$ on $\Gamma_2$ and $w = 0$ on $\partial \Omega_{12} \setminus \Gamma_2$. The solution of this problem in $\Omega_{12}$ is identically null. However, since $\int_{\Omega_{12}} q = \int_{\Omega_{12}} (p_1 - p_2)$ with $p_1 \in Q_{1,0}$ and $p_2$ uniquely determined by the Neumann boundary condition on $\Gamma_N$, we cannot guarantee that $q = 0$.

Notice that a result similar to Proposition 2.3 could be obtained by switching the role of the interface conditions (3) and (5), i.e., considering

$$-\text{div} \ T(u_i, p_i) = f \quad \text{in } \Omega_i, \ i = 1, 2,$$

$$\text{div} \ u_i = 0 \quad \text{in } \Omega_i, \ i = 1, 2,$$

$$u_i = \phi_D \quad \text{on } \Gamma_D^i, \ i = 1, 2,$$

$$T(u_i, p_i) \cdot n = \phi_N \quad \text{on } \Gamma_N^i, \ i = 1, 2,$$

$$u_1 = u_2 \quad \text{on } \Gamma_2.$$

3. FORMULATION OF THE ICDD METHOD FOR THE STOKES PROBLEM

For the sake of simplicity we will consider homogeneous boundary conditions, i.e., we will set $\phi_D = 0$ on $\Gamma_D$ and $\phi_N = 0$ on $\Gamma_N$. Moreover, since we will be interested in computing a finite dimensional approximation of the solution of the Stokes problem, we introduce the ICDD method directly at the discrete level.

3.1. *hp*-FEM discretization

We introduce two regular computational grids $T_1$ and $T_2$ in $\Omega_1$ and $\Omega_2$ made by either simplices or quadrilaterals/hexahedra. We assume that they coincide in $\Omega_1$ and that both interfaces $\Gamma_1$ and $\Gamma_2$ do not cross any element of $\Omega_1$ or $\Omega_2$. We discretize both primal and dual problems in each subdomain by *hp* finite element methods (*hp*-FEM). Because of the difficulty to compute integrals exactly for large $p$, typically when quadrilaterals are used, Legendre-Gauss-Lobatto quadrature formulas are employed to approximate the bilinear forms $a(p)$ and $b(p)$ (see (3)-(6)) as well as the $L^2$-inner products in $\Omega_i$ and on the interfaces. This leads to the so called *Galerkin approach with Numerical Integration* (G-NI) and to the Spectral Element Method with Numerical Integration (SEM-NI). In particular, we consider either inf-sup stable finite dimensional spaces or stabilized couples of spaces of the same degree (see [? ? ? ? ? ] ) to approximate the velocity and the pressure and we assume that the polynomials used for the pressure are continuous (see, e.g., [? ? ? ] ). More precisely, given an integer $p \geq 1$, let $P_p$ be the space of polynomials whose global degree is less than or equal to $p$ in the variables $x_1, \ldots, x_d$ and $Q_p$, be the space of polynomials that are of degree less than or equal to $p$ with respect to each variable $x_1, \ldots, x_d$. The space $P_p$ is associated to simplicial partitions, while $Q_p$ to quadrilateral ones. We introduce the finite dimensional space on $\Omega_i$ defined by

$$X_{l,h} = \{ v \in C^0(\overline{\Omega}) : v|_T \in P_p \ \forall T \in T_i \}$$

in the simplicial case, and by

$$X_{l,h} = \{ v \in C^0(\overline{\Omega}) : v|_T \in Q_p \ \forall T \in T_i \}$$
for quadrilaterals. Then, the finite dimensional spaces for velocity and pressure are, respectively,

\[ V_{i,h} = [X_{i,h}^p]^d \cap V_{i,0}, \quad Q_{i,h} = X_{i,h}^r \]

for suitable polynomial degrees \( p \) and \( r \).

### 3.2. ICDD method with Dirichlet controls

Assume, for simplicity, that \( \partial \Omega_{12} \cap \Gamma_N \neq \emptyset \) and \( \Gamma_D \neq \emptyset \). (We will discuss this issue more in details in section 5.) We define the space of discrete Dirichlet controls as

\[ \mathbf{A}_{i,h}^D = \{ \lambda_{i,h} \in C^0(\Gamma_i) : \exists v_{i,h} \in V_{i,h} \text{ with } \lambda_{i,h} = v_{i,h}|\Gamma_i \} \]

and let

\[ \mathbf{A}_{i,h}^D = \mathbf{A}_{i,h}^D \times \mathbf{A}_{2,h}^D. \]

For \( i = 1, 2 \), we consider two control functions \( \lambda_{i,h} \in \mathbf{A}_{i,h}^D \) and the state problems: find \( (u_{i,h}, p_{i,h}) \in V_{i,h} \times Q_{i,h} \) such that, for all \( (v_{i,h}, q_{i,h}) \in V_{i,h} \times Q_{i,h} \), \( v_{i,h} = 0 \) on \( \Gamma_i \),

\[ a_i(u_{i,h}, v_{i,h}) + b_i(p_{i,h}, v_{i,h}) = \int_{\Gamma_i} f \cdot v_{i,h} \]

\[ b_i(q_{i,h}, u_{i,h}) = 0 \]

where \( a_i \) and \( b_i \) denote the restriction of the bilinear forms \( a \) and \( b \), respectively. However, we do not explicitly express such dependence for the sake of notation.

The unknown controls on the interface are obtained by solving a minimization problem defined through a suitable cost functional depending on the difference between unknown controls on \( \Gamma_1 \) and \( \Gamma_2 \) measured in a suitable norm. More precisely, inspired by (4), we look for

\[ \inf_{\lambda_{i,h}} \left[ J_0(\lambda_{i,h}) : = \frac{1}{2} \| u_{1,h} - u_{2,h} \|^2 \right] \]

To the minimization problem (17) we can associate the following optimality system: find \( \lambda_{i,h} \in \mathbf{A}_{i,h}^D \) and, for \( i = 1, 2 \) \( (u_{i,h}, p_{i,h}) \in V_{i,h} \times Q_{i,h} \), \( (w_{i,h}, q_{i,h}) \in V_{i,h} \times Q_{i,h} \) such that, for all \( (v_{i,h}, q_{i,h}) \in V_{i,h} \times Q_{i,h} \) with \( v_{i,h} = 0 \) on \( \Gamma_i \),

\[ a_i(u_{i,h}, v_{i,h}) + b_i(p_{i,h}, v_{i,h}) = \int_{\Gamma_i} f \cdot v_{i,h} \]

\[ b_i(q_{i,h}, u_{i,h}) = 0 \]

\[ w_{i,h} = (-1)^{i+1}(u_{1,h} - u_{2,h}) \text{ on } \Gamma_i \]

and, for all \( (\mu_{i,h}, \mu_{2,h}) \in \mathbf{A}_{i,h}^D \),

\[ \int_{\Gamma_1} ((u_{1,h} - u_{2,h}) + w_{2,h}) \mu_{1,h} d\Gamma + \int_{\Gamma_2} ((u_{1,h} - u_{2,h}) + w_{1,h}) \mu_{2,h} d\Gamma = 0. \]

### 3.3. Algebraic formulation of ICDD with Dirichlet controls

To the Stokes problem in subdomain \( \Omega_i \) \( (i = 1, 2) \) we can associate the matrix

\[ S_i = \left[ \begin{array}{cc} A_i & B_i^T \\ B_i & 0 \end{array} \right] \]

where \( A_i \) corresponds to the finite dimensional approximation of the bilinear form \( a_{\Omega_i} \) (see (2)), while \( B_i \) corresponds to the discretization of \( b_{\Omega_i} \) (see (3)). When stabilization is used, the matrices \( S_i \) take the form

\[ S_i = \left[ \begin{array}{cc} A_i & B_i^T \\ B_i & 0 \end{array} \right] + \left( \begin{array}{c} \tilde{A}_i \\ \tilde{B}_i \end{array} \right) \tilde{C}_i \]

where \( \tilde{A}_i, \tilde{B}_i \) and \( \tilde{C}_i \) are assembled locally, element by element, and they take into account the integration of the differential operators.

In the following we will denote by the index \( I_i \), the degrees of freedom for the velocity and the pressure belonging to \( \Omega_i \setminus \Gamma_i \), while the index \( I_{\Gamma} \) will refer to the degrees of freedom on the interface \( \Gamma_i \). For the sake of exposition, we will reorder the nodes in \( I_1 \) putting those associated to \( \Omega_1 \setminus \Gamma_1 \) first followed by those on the interfaces. Correspondingly, with obvious choice of notation, we can rewrite the Stokes matrix \( S_i \) as

\[ S_i = \left[ \begin{array}{ccc} A_{I_1, I_1} & B_{I_1, I_1}^T & A_{I_1, I_{\Gamma}} \\ B_{I_1, I_1} & 0 & B_{I_1, I_{\Gamma}} \end{array} \right] \left[ \begin{array}{cc} A_{I_{\Gamma}, I_{\Gamma}} & B_{I_{\Gamma}, I_{\Gamma}}^T \\ B_{I_{\Gamma}, I_{\Gamma}} & 0 \end{array} \right] \]

\[ = \left( \begin{array}{c} S_{I_1, I_1} \\ S_{I_1, I_{\Gamma}} \end{array} \right) \left( \begin{array}{c} S_{I_{\Gamma}, I_{\Gamma}} \\ S_{I_{\Gamma}, I_{1}} \end{array} \right). \]

Moreover, we will indicate by \( M_{I_{\Gamma}} \), the mass matrix on the interface \( \Gamma_{\Gamma} \).

Finally, in the rest of the section, we will denote by \( F_i \) the right-hand side for the state problems in \( \Omega_i \), while \( U_i \) and \( W_i \) will be the vectors of unknown velocity and pressure in \( \Omega_i \) for the state and the adjoint problems, respectively. \( \lambda_{I_i} \) is the vector of the unknown Dirichlet controls on \( \Gamma_i \):

\[ \lambda_{I_i} = \left( (\lambda_{I_i})_1, \ldots, (\lambda_{I_i})_{N_{\Gamma_i}} \right), \quad (\lambda_{I_i})_j = \lambda_{I_i}(x_j), \quad j \in G_i, \]

where \( G_i \) is the set of the \( N_{\Gamma_i} \) indices corresponding to the velocity degrees of freedom on the interface \( \Gamma_i \) and \( x_j \) is a node on \( \Gamma_i \) \( (\lambda_{I_i})_j \) is the nodal value of the discrete control function \( \lambda_{I_i} \) at the node \( x_j \).

We consider now the optimality system associated to the functional \( J_0 \) with Dirichlet controls that we introduced in section 5.3.1. If \( R_{ij} \) denotes the algebraic restriction operator of the velocity unknowns in \( \Omega_i \) to the interface \( \Gamma_j \) \( (i, j = 1, 2) \), the algebraic counterpart of (18) reads

\[ S_i \lambda_{I_i} = b_i \]
where \( y_t = (U_{1,t}, U_{2,t}, W_{1,t}, W_{2,t}, \lambda_{1,t}, \lambda_{2,t})^T \), \( b_t = (F_1, F_2, 0, 0, 0, 0)^T \) and the matrix \( S_t \) is defined as

\[
\begin{pmatrix}
S_{i,i} & 0 & 0 & 0 & S_{i,1} & 0 \\
0 & S_{i,i} & 0 & 0 & S_{i,2} & 0 \\
0 & 0 & S_{i,i} & 0 & S_{i,1} & 0 \\
-2S_{i,i} & 0 & 0 & S_{i,i} & 0 & 0 \\
0 & -2S_{i,i} & 0 & 0 & S_{i,i} & 0 \\
0 & 0 & -2S_{i,i} & 0 & S_{i,i} & 0 \\
\end{pmatrix}
\]

For the numerical solution of the linear systems (21), we compute the Schur complement system with respect to the control variables \((\lambda_{1,i}, \lambda_{2,i})\) and solve them through an iterative method like, e.g., BiCGstab ([2?]).

The Schur complement system reads

\[
\Sigma_t \left( \begin{array}{c}
\lambda_{1,i} \\
\lambda_{2,i}
\end{array} \right) = \chi_t
\]  
(22)

where

\[
\Sigma_t = \begin{pmatrix}
M_{1,1} &=& (R_{1,1} - (R_{1,1}^{-1} S_{i,1} R_{1,1}^{-1}), S_{i,2}) \\
M_{1,2} &=& (R_{1,2} - (R_{1,2}^{-1} S_{i,2} R_{1,2}^{-1}), S_{i,1})
\end{pmatrix}
\]

and

\[
\chi_t = \begin{pmatrix}
M_{1,1} R_{1,1} &=& (R_{1,1}^{-1} S_{i,1} R_{1,1}^{-1}, S_{i,2}) \\
M_{1,2} R_{1,2} &=& (R_{1,2}^{-1} S_{i,2} R_{1,2}^{-1}, S_{i,1})
\end{pmatrix}
\]

\( I_{\Gamma_i} \) is the identity matrix on the interface \( \Gamma_i \).

### 3.4. ICDD method with Neumann and mixed controls

Let \( \Lambda_{N,h}^i \) denote the space of discrete Neumann controls on \( \Gamma_i \). We require that \( \Lambda_{N,h}^i \subset L^2(\Gamma_i) \).

For \( i = 1, 2 \), given the control functions \( \lambda_{i,h} \in \Lambda_{N,h}^i \) and the discrete state problems: find \((u_{i,h}, p_{i,h}) \in V_{i,h} \times Q_{i,h} \), such that, for all \((v_{i,h}, q_{i,h}) \in V_{i,h} \times Q_{i,h}\),

\[
a_i(u_{i,h}, v_{i,h}) + b_i(p_{i,h}, v_{i,h}) = \int_{\Gamma_i} \lambda_{i,h} \cdot v_{i,h} \\
+ \int_{\Omega_i} f \cdot v_{i,h} \\
b_i(q_{i,h}, u_{i,h}) = 0.
\]

Let \( T_k \subset \Omega_i \) be a generic element in \( \Omega_i \); we introduce the set \( E_k = \{ k : \text{meas}(\partial T_k \cap \Gamma_i) > 0 \} \) and, for any \( k \in E_k \), the edges \( e_{ik} = \partial T_k \cap \Gamma_i \). Thanks to the definition of \([X_{P,h}^{\ell}]^d\), for any \( v_{i,h} \in [X_{P,h}^{\ell}]^d \) and \( q_{i,h} \in X_{P,h}^{\ell} \), it holds \( v_{i,h} \cdot n_{i,h} \) \( \in \{C^1(\overline{T_k})\}^d \) and \( q_{i,h} \cdot n_{i,h} \in C^0(\overline{T_k}) \), and then we define the discrete normal stress

\[
\Phi_{i,h} = T(u_{i,h}, p_{i,h}) \cdot n \quad \text{on} \quad \Gamma_i.
\]

This definition makes sense in classical way on each \( e_{ik} \subset \Gamma_i \), so that \( \Phi_{i,h} \in [L^2(\Gamma_i)]^d \).

We are interested in evaluating the discrete normal stress associated to \((u_{i,h}, p_{i,h})\) also on the interface \( \Gamma_j (j = 3 - i) \), which is internal to \( \Omega_i \).

To this aim we restrict \((u_{i,h}, p_{i,h})\) to \( \Omega_{12} \) and then we extend it to \( \Omega_i \) so that such extension \((\tilde{u}_{i,h}, \tilde{p}_{i,h})\) belongs to \( V_{i,h} \times Q_{i,h} \). Then we define

\[
\Phi_{i,j,h} = T(\tilde{u}_{i,h}, \tilde{p}_{i,h}) \cdot n \quad \text{on} \quad \Gamma_j
\]

and it holds \( \Phi_{i,j,h} \in [L^2(\Gamma_j)]^d \).

Following (5), the discrete Neumann controls \( \lambda_{i,h} \) on the interface \( \Gamma_i \) are obtained as solution of the following minimization problem

\[
\inf_{\lambda_{1,h}, \lambda_{2,h}} \{ J_f(\lambda_{1,h}, \lambda_{2,h}) = \frac{1}{2} \sum_{i=1}^2 \| \Phi_{i,1,h} - \Phi_{i,2,h} \|_{L^2(\Gamma_i)}^2 \}.
\]

In practice, the discrete normal stresses on the interfaces \( \Gamma_i \) are obtained as residuals of the first equation in (22), see below.

Let \( T_i^\ell \) and \( T_i^p \) be the sets of indices of the nodes of the meshes in \( \Omega_i \) for the velocity and the pressure, respectively. Moreover, let \( G_{i,\nu}^{\ell} \subset T_i^\ell \) be the subsets of indices of the nodes lying on \( \Gamma_i \). We consider matching meshes on the overlap \( \Omega_{12} \). In \([X_{P,h}^{\ell}]^d\) we take the basis \( B_i^{\ell} \) of the characteristic Lagrange polynomials \( \varphi_{i,\ell} \) with \( \ell \in T_i^\ell \). Similarly, in \( Q_{i,h} \) we consider the basis \( B_i^{\ell} \) of the characteristic Lagrange polynomials \( \psi_{i,\ell} \), with \( k \in T_i^p \).

Now, let \((u_{i,h}, p_{i,h})\) be the solution of (23). For any \( \ell \in G_{i,\nu}^{\ell} \), we define the vectors \((\Phi_{i,\ell}, \varphi_{i,\ell}) \in \mathbb{R}^d\) of the weak discrete normal stresses on \( \Gamma_i \) associated to \((u_{i,h}, p_{i,h})\) as

\[
(\Phi_{i,\ell}, \varphi_{i,\ell}) = a_i(u_{i,h}, \varphi_{i,\ell}) + b_i(p_{i,h}, \varphi_{i,\ell}) = \int_{\Gamma_i} \lambda_{i,h} \cdot \varphi_{i,\ell} \\
+ \int_{\Omega_i} f \cdot \varphi_{i,\ell}.
\]

Similarly, for any \( \ell \in G_{i,\nu}^{\ell} \) and \( \varphi_{i,\ell} \in \mathbb{R}^d \) we define the vectors \((\Phi_{i,\ell}, \varphi_{i,\ell}) \in \mathbb{R}^d\) of the weak discrete normal stresses on \( \Gamma_j \) associated to \((u_{i,h}, p_{i,h})\) as

\[
(\Phi_{i,\ell}, \varphi_{i,\ell}) = a_j(u_{i,h}, \varphi_{i,\ell}) + b_j(p_{i,h}, \varphi_{i,\ell}) = \int_{\Omega_j} \varphi_{i,\ell}.
\]

It holds

\[
(\Phi_{i,\ell}, \varphi_{i,\ell}) = \int_{\Gamma_{ij}} \Phi_{i,j,h} \cdot \varphi_{i,\ell} \quad \forall \ell \in G_{i,\nu}^{\ell}, \quad i,j \in \{1, 2\}.
\]

To the minimization of problem (24) we can associate the following optimality system: find \( \lambda_{N,h} = (\lambda_{1,N}, \lambda_{2,N}) \in \Lambda_{N,h} \) and, for \( i = 1, 2 \), \((u_{i,h}, p_{i,h}), (w_{i,h}, q_{i,h}) \in V_{i,h} \times Q_{i,h}\) such that

\[
a_i(u_{i,h}, \varphi_{i,\ell}) + b_i(p_{i,h}, \varphi_{i,\ell}) = \int_{\Gamma_i} \lambda_{i,h} \cdot \varphi_{i,\ell} \\
+ \int_{\Omega_i} f \cdot \varphi_{i,\ell} \quad \forall \ell \in T_i^\ell,
\]

\[
b_i(q_{i,h}, u_{i,h}) = 0 \quad \forall k \in T_i^p.
\]
\[ a_i(w_{i,h}, \varphi_{i,\ell}) + b_i(q_{i,h}, \varphi_{i,\ell}) = (\Phi_{i,\Gamma_i})_\ell - (\Phi_{i,\Gamma_i})_\ell \quad \forall \ell \in \mathcal{I}_h^1 \tag{28} \]

\[ b_i(p_{i,h}, w_{i,h}) = 0 \quad \forall k \in \mathcal{I}_h^1, \]

and

\[ \sum_{i=1}^2 [(\Phi_{i,\Gamma_i})_\ell - (\Phi_{i,\Gamma_i})_\ell + (\Psi_{i,\Gamma_i})_\ell] = 0 \quad \forall \ell \in \mathcal{G}_h^\Gamma, \tag{29} \]

where \( j = 3 - i \) and

\[ (\Psi_{j,\Gamma_i})_\ell = a_j(w_{i,h}, \varphi_{j,\ell}) + b_j(q_{i,h}, \varphi_{j,\ell}) \]

is the weak discrete normal stress on \( \Gamma_j \) associated to the dual state solution \((w_{i,h}, q_{i,h})\).

An alternative strategy consists in choosing mixed Dirichlet control \( \lambda_1,h \in \mathcal{A}_{1,h}^D \) on \( \Gamma_1 \) and a Neumann control \( \lambda_2,h \in \mathcal{A}_{2,h}^N \) on \( \Gamma_2 \) to minimize the difference between both interface velocities and interface normal stresses.

Following \( 65 \) and \( 66 \), the corresponding minimization problems would read:

\[
\inf_{\lambda_1,h, \lambda_2,h} \left[ J_{f1}(\lambda_1,h, \lambda_2,h) := \frac{1}{2} \| u_{1,h} - u_{2,h} \|^2_{L^2(\Gamma_1)} + \frac{1}{2} \| \Phi_{1,2,h} - \Phi_{2,2,h} \|^2_{L^2(\Gamma_2)} \right] \tag{30}
\]

Alternatively, following \( 145 \) and \( 146 \), we could consider a discrete Neumann control on \( \Gamma_1 \) and a discrete Dirichlet control on \( \Gamma_2 \) and the corresponding minimization problem:

\[
\inf_{\lambda_1,h, \lambda_2,h} \left[ J_{f2}(\lambda_1,h, \lambda_2,h) := \frac{1}{2} \| \Phi_{1,1,h} - \Phi_{2,1,h} \|^2_{L^2(\Gamma_1)} + \frac{1}{2} \| u_{1,h} - u_{2,h} \|^2_{L^2(\Gamma_2)} \right] \tag{31}
\]

To the minimization problem \( 31 \) we associate the following optimality system: find \( \lambda_h = (\lambda_{1,h}, \lambda_{2,h}) \in \mathcal{A}_{1,h}^D \times \mathcal{A}_{2,h}^N \) and, for \( i = 1, 2, (u_i,h, p_i,h) \in \mathcal{V}_{i,h} \times \mathcal{Q}_{i,h}, (w_i,h, q_i,h) \in \mathcal{V}_{i,h} \times \mathcal{Q}_{i,h} \) such that

\[
a_1(u_{i,h}, \varphi_{i,\ell}) + b_1(p_{i,h}, \varphi_{i,\ell}) = \int_{\Omega_i} f \cdot \varphi_{i,\ell}, \forall \ell \in \mathcal{I}_h^1 \tag{32} \]

\[ b_1(p_{i,h}, u_{i,h}) = 0 \quad \forall k \in \mathcal{I}_h^1, \]

\[ u_{1,h} = \lambda_{1,h} \quad \text{on } \Gamma_1 \]

\[
a_2(u_{2,h}, \varphi_{2,\ell}) + b_2(p_{2,h}, \varphi_{2,\ell}) = \int_{\Omega_2} \lambda_{2,h} \cdot \varphi_{2,\ell} + \int_{\Gamma_2} f \cdot \varphi_{2,\ell}, \forall \ell \in \mathcal{I}_h^2 \tag{33} \]

\[ b_2(p_{2,h}, u_{2,h}) = 0 \quad \forall k \in \mathcal{I}_h^2, \]

\[ a_1(w_{1,h}, \varphi_{1,\ell}) + b_1(q_{1,h}, \varphi_{1,\ell}) = 0 \quad \forall \ell \in \mathcal{I}_h^1 \tag{34} \]

\[ b_1(p_{1,h}, w_{1,h}) = 0 \quad \forall k \in \mathcal{I}_h^1, \]

\[ w_{1,h} = u_{1,h} - u_{2,h} \quad \text{on } \Gamma_1 \]

\[
a_2(w_{2,h}, \varphi_{2,\ell}) + b_2(q_{2,h}, \varphi_{2,\ell}) = (\Phi_{2,\Gamma_2})_\ell - (\Phi_{2,\Gamma_2})_\ell \quad \forall \ell \in \mathcal{I}_h^2 \tag{35} \]

\[ b_2(p_{2,h}, w_{2,h}) = 0 \quad \forall k \in \mathcal{I}_h^2, \]

and

\[
\left[(w_{1,h}|_{\Gamma_1})_\ell - (w_{2,h}|_{\Gamma_1})_\ell + (w_{2,h} |_{\Gamma_1})_\ell \right] \quad \forall \ell \in \mathcal{G}_h^\Gamma, \forall j \in \mathcal{G}_h^\Gamma \tag{36} \]

Similarly, to the minimization problem \( 31 \) we associate the optimality system: find \( \Lambda_h = (\lambda_{1,h}, \lambda_{2,h}) \in \mathcal{A}_{1,h}^N \times \mathcal{A}_{2,h}^D \) and, for \( i = 1, 2, (u_i,h, p_i,h) \in \mathcal{V}_{i,h} \times \mathcal{Q}_{i,h}, (w_i,h, q_i,h) \in \mathcal{V}_{i,h} \times \mathcal{Q}_{i,h} \) such that

\[
a_1(u_{1,h}, \varphi_{1,\ell}) + b_1(p_{1,h}, \varphi_{1,\ell}) = \int_{\Omega_1} \lambda_{1,h} \cdot \varphi_{1,\ell} + \int_{\Omega_1} f \cdot \varphi_{1,\ell}, \forall \ell \in \mathcal{I}_h^1 \tag{37} \]

\[ b_1(p_{1,h}, u_{1,h}) = 0 \quad \forall k \in \mathcal{I}_h^1, \]

\[ u_{2,h} = \lambda_{2,h} \quad \text{on } \Gamma_2 \]

\[
\left[(u_{1,h}|_{\Gamma_2})_\ell - (u_{2,h}|_{\Gamma_2})_\ell + (u_{2,h} |_{\Gamma_2})_\ell \right] \quad \forall j \in \mathcal{G}_h^\Gamma, \forall \ell \in \mathcal{G}_h^\Gamma \tag{38} \]

\[
\left[(\Phi_{1,\Gamma_1})_\ell - (\Phi_{2,\Gamma_1})_\ell + (\Phi_{2,\Gamma_1})_\ell \right] \quad \forall \ell \in \mathcal{G}_h^\Gamma, \forall k \in \mathcal{G}_h^\Gamma \tag{39} \]

\[
\left[(\Phi_{1,\Gamma_2})_\ell - (\Phi_{2,\Gamma_2})_\ell + (\Phi_{2,\Gamma_2})_\ell \right] \quad \forall \ell \in \mathcal{G}_h^\Gamma, \forall k \in \mathcal{G}_h^\Gamma \tag{40} \]

\[ w_{2,h} = u_{1,h} - u_{2,h} \quad \text{on } \Gamma_2 \]

3.5. Algebraic formulation of ICDD with Neumann and mixed controls

Using the previous notations, the discrete values of the Neumann controls are given by

\[ (\lambda_{i,h})_\ell = \sum_{k \in E_i \cap E_{ik}} \lambda_{i,h} \cdot \varphi_{i,\ell} \quad \forall \ell \in \mathcal{G}_h^\Gamma. \]
We report the number of iterations required to converge, the computed infimum of the cost functional $J_f$ and the errors

$$
epsilon^m_{1} = \left( \|u_1 - u_{1,h}\|_{H^1(\Omega_1)} + \|u_2 - u_{2,h}\|_{H^1(\Omega_2)} \right)^{1/2},$$

$$
epsilon^m_{2} = \left( \|p_1 - p_{1,h}\|_{L^2(\Omega_1)} + \|p_2 - p_{2,h}\|_{L^2(\Omega_2)} \right)^{1/2},$$

$$
epsilon^m_{1,2,h} = \|u_{1,h} - u_{2,h}\|_{L^2(\Omega_{1,2})},$$

where $u_{1,h}, u_{2,h} \in V_{1,h}$ and $p_{1,h}, p_{2,h} \in Q_{1,h}$ are the discrete solutions approximating the solutions of (59)-(60).

### 4. Numerical Results

#### 4.1. Test cases with respect to an analytic solution

We consider the domain $\Omega = (0, 1) \times (0, 2)$ with $\Omega_1 = (0, 1) \times (1 - \delta/2, 2)$ and $\Omega_2 = (0, 1) \times (0, 1 + \delta/2)$, $\delta > 0$ being a suitable parameter characterizing the width of the overlapping region. The viscosity $\nu$ is set to 1, while the force $f$ and the boundary conditions are chosen such that the Stokes problem admits the solution $u = (\exp(y), -\exp(x))^T$ and $p = \exp(x)\sin(y)$. Concerning the boundary conditions, we impose Neumann conditions on the boundary $1 \times (0, 2)$ while Dirichlet boundary conditions are imposed on the remaining boundaries. We compute the solution of the optimality system using the BiCGStab method on the Schur complement (22) setting the tolerance to $10^{-9}$.

First, we consider the case of an overlap with fixed width $\delta = 0.2$. We use both Taylor-Hood elements with three computational meshes characterized by $h = 2^{-2}, 2^{-3}, 2^{-4}$, and stabilized $hp$-FEM $Q_p - Q_p$ [7]. In the latter case, we consider 4 × 5 quadrature elements in each subdomain $\Omega_1, 4 \times 1$ elements in $\Omega_{1,2}$ and each quadrilateral element has sides of length $h = 2^{-2}$.

In tables II and III we report the number of iterations required to converge, the computed infimum of the cost functional $J_f$ and the errors $\epsilon^m_{1} = \left( \|u_1 - u_{1,h}\|_{H^1(\Omega_1)} + \|u_2 - u_{2,h}\|_{H^1(\Omega_2)} \right)^{1/2}$, $\epsilon^m_{2} = \left( \|p_1 - p_{1,h}\|_{L^2(\Omega_1)} + \|p_2 - p_{2,h}\|_{L^2(\Omega_2)} \right)^{1/2}$, $\epsilon^m_{1,2,h} = \|u_{1,h} - u_{2,h}\|_{L^2(\Omega_{1,2})}$, where $u_{1,h}, u_{2,h} \in V_{1,h}$ and $p_{1,h}, p_{2,h} \in Q_{1,h}$ are the discrete solutions approximating the solutions of (59)-(60).

### Table I: Test case with analytic solution. Results for the functional $J_f$ with Taylor-Hood elements with respect to different values of $h$. Fixed overlap with $\delta = 0.2$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>#iter</th>
<th>$\inf J_f$</th>
<th>$\epsilon^m_{1}$</th>
<th>$\epsilon^m_{2}$</th>
<th>$\epsilon^m_{1,2,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{-2}$</td>
<td>8</td>
<td>1.047e-16</td>
<td>1.146e-02</td>
<td>9.322e-03</td>
<td>8.081e-05</td>
</tr>
<tr>
<td>$2^{-3}$</td>
<td>9</td>
<td>8.612e-20</td>
<td>2.835e-03</td>
<td>9.175e-03</td>
<td>6.033e-06</td>
</tr>
<tr>
<td>$2^{-4}$</td>
<td>9</td>
<td>6.003e-20</td>
<td>7.088e-04</td>
<td>5.345e-04</td>
<td>5.432e-07</td>
</tr>
</tbody>
</table>

### Table II: Test case with analytic solution. Results for the functional $J_f$ with stabilized $Q_p - Q_p$ elements with respect to different polynomial degrees $p$. Fixed overlap with $\delta = 0.2$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>#iter</th>
<th>$\inf J_f$</th>
<th>$\epsilon^m_{1}$</th>
<th>$\epsilon^m_{2}$</th>
<th>$\epsilon^m_{1,2,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>1.759e-19</td>
<td>1.402e-03</td>
<td>2.676e-03</td>
<td>8.471e-06</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>1.402e-19</td>
<td>6.719e-05</td>
<td>1.299e-04</td>
<td>6.220e-07</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>3.766e-20</td>
<td>3.922e-07</td>
<td>3.157e-07</td>
<td>5.870e-10</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>1.308e-20</td>
<td>8.740e-09</td>
<td>1.269e-08</td>
<td>7.617e-11</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>1.296e-20</td>
<td>1.129e-09</td>
<td>2.020e-09</td>
<td>5.011e-11</td>
</tr>
</tbody>
</table>
We can see that the number of iterations is independent of both the grid size $h$ and the polynomial degree $p$.

Next, we study the case where the width of the overlap tends to zero on a fixed computational mesh. When using the Taylor-Hood elements, we set $h = 0.04$ and $\delta = 5h, \ldots, h$; the subdomains are defined as follows: for $\delta = 5h$, $\Omega_1 = (0,1) \times (0,0.92,2)$ and $\Omega_2 = (0,1) \times (0,1,1,12)$; for $\delta = 4h$, $\Omega_1 = (0,1) \times (0,0.92,2)$ and $\Omega_2 = (0,1) \times (0,1,1,08)$; for $\delta = 3h$, $\Omega_1 = (0,1) \times (0,0.96,2)$ and $\Omega_2 = (0,1) \times (0,1,04)$; for $\delta = 2h$, $\Omega_1 = (0,1) \times (0,0.96,2)$ and $\Omega_2 = (0,1) \times (0,1,04)$; for $\delta = 4p$, $\Omega_1 = (0,1) \times (0,0.96,2)$ and $\Omega_2 = (0,1) \times (0,1,04)$. For stabilized $Q_p - Q_p$, we take $p = 4$ and we partition each subdomain in $4 \times 4$ quad elements; $\Omega_1 \setminus \Omega_1$ is partitioned into $4 \times 4$ equal quad elements of size $h_x \times h_y$. $h_x = 0.25$ and $h_y = (1 - \delta/2)/4$; $\Omega_1$ is partitioned in $1 \times 5$ quads of size $h_x \cdot \delta$; the value of $\delta$ ranges from 0.2 to 0.01. Results reported in tables [III] and [IV] show that the required number of iterations increases when $\delta$ decreases.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\text{iter}$</th>
<th>$\inf J_1$</th>
<th>$e_1^h$</th>
<th>$e_0^h$</th>
<th>$e_{12}^h$</th>
<th>$e_{12}^h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4h$</td>
<td>10</td>
<td>1.565e-16</td>
<td>3.991e-04</td>
<td>2.342e-04</td>
<td>2.553e-07</td>
<td>3.611e-05</td>
</tr>
<tr>
<td>$3h$</td>
<td>13</td>
<td>8.237e-19</td>
<td>3.967e-04</td>
<td>2.312e-04</td>
<td>2.418e-07</td>
<td>4.456e-05</td>
</tr>
</tbody>
</table>

Finally, we carry out a convergence test with Taylor-Hood elements setting $\delta = h$ and letting $h \to 0$. Also in this case we can see that the number of iterations required to converge grows when $h$ decreases. Results are reported in table [V].

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\text{iter}$</th>
<th>$\inf J_1$</th>
<th>$e_1^h$</th>
<th>$e_0^h$</th>
<th>$e_{12}^h$</th>
<th>$e_{12}^h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>9</td>
<td>3.766e-20</td>
<td>3.922e-07</td>
<td>3.157e-07</td>
<td>5.870e-10</td>
<td>8.652e-08</td>
</tr>
<tr>
<td>0.1</td>
<td>15</td>
<td>2.391e-17</td>
<td>5.956e-07</td>
<td>7.930e-07</td>
<td>1.729e-09</td>
<td>4.741e-07</td>
</tr>
<tr>
<td>0.05</td>
<td>25</td>
<td>1.266e-17</td>
<td>2.808e-06</td>
<td>5.088e-06</td>
<td>1.333e-09</td>
<td>2.561e-06</td>
</tr>
<tr>
<td>0.02</td>
<td>71</td>
<td>3.369e-16</td>
<td>2.699e-05</td>
<td>4.974e-05</td>
<td>2.856e-09</td>
<td>1.571e-05</td>
</tr>
<tr>
<td>0.01</td>
<td>250</td>
<td>4.208e-04</td>
<td>1.614e+01</td>
<td>2.909e+01</td>
<td>2.056e-03</td>
<td>7.755e+00</td>
</tr>
</tbody>
</table>

These numerical results show that the ICDD method is not very effective especially when considering small overlapping regions. This behavior may due to the fact that the functional $J_f$ involves no information on the pressure fields in the overlap, since it imposes only the continuity of velocities on the interfaces.

The number of iterations is independent of the mesh size $h$ and of the polynomial degree $p$. However, a dependence on the size of the overlap can be estimated as

$$\text{#iter} \sim C \delta^{-1},$$

for a suitable positive constant $C > 0$.

We consider now the case of Neumann and mixed controls.

First, we consider the case of an overlap with fixed width $\delta = 0.2$. The setting and the discretization are the same used before. In tables [VI] and [VII] we report the number of iterations and the computed errors for the case of the functional $J_f$ using Taylor-Hood and stabilized $Q_p - Q_p$, respectively, while in tables [VIII] and [IX] we report the results obtained for the functional $J_f$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\text{iter}$</th>
<th>$\inf J_1$</th>
<th>$e_1^h$</th>
<th>$e_0^h$</th>
<th>$e_{12}^h$</th>
<th>$e_{12}^h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>1/3</td>
<td>6</td>
<td>4.687e-18</td>
<td>2.615e-02</td>
<td>2.345e-02</td>
<td>5.390e-04</td>
</tr>
<tr>
<td>1/6</td>
<td>10</td>
<td>1.309e-17</td>
<td>6.779e-03</td>
<td>5.792e-03</td>
<td>3.262e-05</td>
<td>5.305e-03</td>
</tr>
<tr>
<td>2/5</td>
<td>19</td>
<td>4.989e-16</td>
<td>1.626e-03</td>
<td>1.533e-03</td>
<td>2.240e-06</td>
<td>7.548e-04</td>
</tr>
</tbody>
</table>

Then, we consider the case where the width of the overlap tends to zero on a fixed computational mesh. Results are shown in tables [X] and [XI] for the Taylor-Hood elements with $h = 0.04$ and in tables [X] and [XII] for the stabilized $Q_p - Q_p$ with $p = 4$. Both functionals $J_f$ and $J_{1f}$ are used.

Finally, we study the behavior of the ICDD method with functionals $J_f$ and $J_{1f}$ using Taylor-Hood elements setting $\delta = h$ and letting $h \to 0$. Results are reported in tables [XIV] and [XV].

Differently from the case of Dirichlet controls with functional $J_f$, we can see that both functionals $J_f$ and $J_{1f}$ require a much lower number of iterations to converge.
TABLE VIII: Test case with analytic solution. Results for the functional $J_{f}$ with Taylor-Hood elements with respect to different values of $h$. Fixed overlap with $\delta = 0.2$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>#iter</th>
<th>$\inf J_{f}$</th>
<th>$e_{l}^{v}$</th>
<th>$e_{0}^{p}$</th>
<th>$e_{12.0}^{v}$</th>
<th>$e_{12.0}^{p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2/2</td>
<td>6</td>
<td>2.731e-03</td>
<td>1.126e-02</td>
<td>8.778e-03</td>
<td>5.597e-05</td>
<td>2.285e-03</td>
</tr>
<tr>
<td>2/3</td>
<td>6</td>
<td>6.964e-04</td>
<td>2.829e-03</td>
<td>2.143e-04</td>
<td>4.352e-06</td>
<td>2.885e-04</td>
</tr>
</tbody>
</table>

TABLE IX: Test case with analytic solution. Results for the functional $J_{f}$ with stabilized $Q_{p} - Q_{p}$ elements with respect to different polynomial degrees $p$. Fixed overlap with $\delta = 0.2$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>#iter</th>
<th>$\inf J_{f}$</th>
<th>$e_{l}^{v}$</th>
<th>$e_{0}^{p}$</th>
<th>$e_{12.0}^{v}$</th>
<th>$e_{12.0}^{p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>4.003e-18</td>
<td>1.231e-03</td>
<td>2.391e-03</td>
<td>1.101e-05</td>
<td>6.378e-04</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>3.044e-18</td>
<td>2.147e-05</td>
<td>5.461e-05</td>
<td>1.894e-07</td>
<td>1.281e-05</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>2.185e-18</td>
<td>6.358e-09</td>
<td>6.231e-09</td>
<td>8.761e-11</td>
<td>2.384e-09</td>
</tr>
</tbody>
</table>

shows that controlling the pressure and not only the velocity on the interfaces is crucial for the Stokes problem.

Moreover, we can see that the best convergence results are obtained with mixed controls and functional $J_{f}$; as a matter of fact, in this case the number of iterations is independent from the mesh size $h$, from the degree $p$ of polynomial used, and from the measure $\delta$ of the overlap.

Neumann controls with functional $J_{f}$ also provide a number of iterations independent of the mesh size $h$ and of the polynomial degree $p$. However, a dependence on the size of the overlap can be noticed as

$$\# \text{iter} \sim C\delta^{-1/2}$$

for a suitable positive constant $C > 0$.

4.2. A test case without analytic solution

We consider the computational domain $\Omega = (0,1) \times (0,2)$ with $\Omega_{1} = (0,1) \times (1 - \delta/2,2)$ and $\Omega_{2} = (0,1) \times (0,1 + \delta/2)$, as represented schematically in Figure 2. The force is set to $f = 0$ and the viscosity is $\nu = 2.0e^{-3}$. We impose homogeneous Neumann boundary conditions on the edges $l_{1}$ and $l_{2}$. On the remaining boundaries, apart from

the edge $l_{6}$, we impose homogeneous Dirichlet boundary conditions unless on $\{0\} \times (1.1,2)$ where we set a parabolic profile with maximum equal to 1.

On the edge $l_{6}$ we may impose either homogeneous Neumann or Dirichlet boundary conditions to compare the behavior of the different methods that we have studied. In particular, we want to show that the functional $J_{f}$ with Dirichlet controls will not provide a correct solution when $l_{6}$ is set as a Dirichlet boundary, since this case violates the Assumption 2.

For this problem, besides the errors $e_{12.0}^{v}$ and $e_{12.0}^{p}$ on the overlap, we also compute

$$e_{l}^{v} = \left( \|U_{1,h} - u_{1,h}\|_{H^{1}(\Omega_{1})} + \|U_{2,h} - u_{2,h}\|_{H^{1}(\Omega_{2})} \right)^{1/2},$$

$$e_{l}^{p} = \left( \|P_{1,h} - p_{1,h}\|_{L^{2}(\Omega_{1})} + \|P_{2,h} - p_{2,h}\|_{L^{2}(\Omega_{2})} \right)^{1/2},$$

where $(U_{i,h},P_{i,h})$ is the restriction to the subdomain $\Omega_{i}$ of the solution computed on the same mesh but considering the domain as a whole without any splitting and solving [1].

First, we impose homogeneous Dirichlet boundary con-

TABLE XI: Test case with analytic solution. Results for the functional $J_{f}$ with stabilized $Q_{p} - Q_{p}$ elements with respect to different polynomial degrees $p$ for $\delta \to 0$. By $*^*$ we denote that the method did not converge within 184 iterations.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>#iter</th>
<th>$\inf J_{f}$</th>
<th>$e_{l}^{v}$</th>
<th>$e_{0}^{p}$</th>
<th>$e_{12,0}^{v}$</th>
<th>$e_{12,0}^{p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>9</td>
<td>2.438e-18</td>
<td>3.899e-07</td>
<td>3.031e-07</td>
<td>3.230e-09</td>
<td>2.777e-08</td>
</tr>
<tr>
<td>0.1</td>
<td>15</td>
<td>3.759e-17</td>
<td>4.635e-07</td>
<td>4.247e-07</td>
<td>8.579e-09</td>
<td>2.059e-08</td>
</tr>
<tr>
<td>0.05</td>
<td>25</td>
<td>4.783e-15</td>
<td>6.835e-07</td>
<td>7.114e-07</td>
<td>5.991e-08</td>
<td>2.168e-08</td>
</tr>
<tr>
<td>0.02</td>
<td>96</td>
<td>2.032e-16</td>
<td>6.653e-07</td>
<td>6.873e-07</td>
<td>3.315e-08</td>
<td>2.328e-08</td>
</tr>
<tr>
<td>0.01</td>
<td>250*</td>
<td>5.751e-04</td>
<td>8.371e-01</td>
<td>9.842e-01</td>
<td>5.206e-02</td>
<td>1.733e-03</td>
</tr>
</tbody>
</table>

TABLE XII: Test case with analytic solution. Results for the functional $J_{f}$ with Taylor-Hood elements with $h = 0.04$ and $\delta \to 0$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>#iter</th>
<th>$\inf J_{f}$</th>
<th>$e_{l}^{v}$</th>
<th>$e_{0}^{p}$</th>
<th>$e_{12,0}^{v}$</th>
<th>$e_{12,0}^{p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3h</td>
<td>7</td>
<td>4.543e-04</td>
<td>3.964e-04</td>
<td>3.206e-04</td>
<td>3.163e-07</td>
<td>2.152e-05</td>
</tr>
<tr>
<td>h</td>
<td>7</td>
<td>4.669e-04</td>
<td>3.907e-04</td>
<td>3.141e-04</td>
<td>1.596e-07</td>
<td>1.332e-05</td>
</tr>
</tbody>
</table>

TABLE X: Test case with analytic solution. Results for the functional $J_{f}$ with Taylor-Hood elements with $h = 0.04$ and $\delta \to 0$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>#iter</th>
<th>$\inf J_{f}$</th>
<th>$e_{l}^{v}$</th>
<th>$e_{0}^{p}$</th>
<th>$e_{12.0}^{v}$</th>
<th>$e_{12.0}^{p}$</th>
</tr>
</thead>
</table>
We show the monodomain solution, while in Figures 6 and 8 we show, respectively, the solutions obtained through minimization of the functional $J_f$ and $J_{f'}$. We can see that, although the functional $J_f$ has no control on the pressure, the Neumann boundary condition on the edge $l_0$ allows the pressure to match almost perfectly in the overlapping region. Notice that the difference shown in Fig. 7 is of the same order of the errors reported in tables XIV and XV.

<table>
<thead>
<tr>
<th>Table XIV: Test case with analytic solution. Results for the functional $J_f$ with Taylor-Hood elements with $\delta = h$ and $\delta \to 0$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = h$</td>
</tr>
<tr>
<td>1/3</td>
</tr>
<tr>
<td>1/6</td>
</tr>
<tr>
<td>2/25</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table XV: Test case with analytic solution. Results for the functional $J_{f'}$ with Taylor-Hood elements with $\delta = h$ and $\delta \to 0$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = h$</td>
</tr>
<tr>
<td>1/3</td>
</tr>
<tr>
<td>1/6</td>
</tr>
<tr>
<td>2/25</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table XVI: Test case without analytic solution. Dirichlet boundary condition on $l_0$. Results for the functionals $J_t$ (top), $J_f$ (mid) and $J_{f'}$ (bottom) with Taylor-Hood elements with fixed $h = 0.04$ and $\delta \to 0$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
</tr>
<tr>
<td>4h</td>
</tr>
<tr>
<td>3h</td>
</tr>
<tr>
<td>h</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table XVII: Test case without analytic solution. Neumann boundary condition on $l_0$. Results for the functionals $J_t$ (top), $J_f$ (mid) and $J_{f'}$ (bottom) with Taylor-Hood elements and $p = 6$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
</tr>
<tr>
<td>5h</td>
</tr>
<tr>
<td>4h</td>
</tr>
<tr>
<td>3h</td>
</tr>
<tr>
<td>2h</td>
</tr>
<tr>
<td>h</td>
</tr>
</tbody>
</table>

FIG. 2: Schematic representation of the computational domain.
TABLE XVII: Test case without analytic solution. Dirichlet boundary condition on \( l_0 \). Results for the functionals \( J_i \) (top), \( J_f \) (mid) and \( J_{1f} \) (bottom) with stabilized \( Q_p - Q_v \) elements with fixed \( p = 6 \) and \( \delta \to 0 \).

![Table XVII](image)

Finally, we want to assess the robustness of the method with respect to the viscosity coefficient \( \nu \). Thus, we compute the solution of the problem with Neumann boundary condition on \( l_0 \) using the ICDD method associated to the functional \( J_{1f} \), which is the one that provided the best results in the previous tests. We consider a discretization by Taylor-Hood elements on a mesh with fixed \( \nu = 0.05 \) and \( \delta \to 0 \) and we set the viscosity \( \nu = 10^{-2} \cdot 10^{-4} \cdot 10^{-6} \). Numerical results are reported in tables [XXI][XXII] and they show that the method is robust with respect to variations of the parameter \( \nu \).

5. ANALYSIS OF THE ICDD METHOD FOR THE STOKES PROBLEM

In this section we analyze the ICDD method that we have presented in the previous sections with the aim of guaranteeing the well-posedness of the minimization problem. We begin with the analysis in the continuous case and for Dirichlet controls.

5.1. Analysis of the optimal control problem with Dirichlet controls

For \( i = 1, 2 \), we introduce the following spaces:

\[
\begin{align*}
\Lambda_i &= \{ \mu \in [H^{1/2}(\Gamma_i)]^d : \exists \nu \in [H^1(\Omega_i)]^d, \mu \cdot v = 0 \text{ on } \Gamma_i^D \} \\
\Lambda_{i,0} &= \{ \mu \in \Lambda_i : \int_{\Gamma_i} \mu \cdot n = 0 \} \\
\Lambda^D &= \Lambda_1^D \times \Lambda_2^D 
\end{align*}
\]

We will denote by

\[
\Lambda^D = \Lambda_1^D \times \Lambda_2^D 
\]

the spaces of admissible Dirichlet controls. Moreover, we will denote

\[
\Lambda^D = \Lambda_1^D \times \Lambda_2^D 
\]
TABLE XI: Test case without analytic solution. Neumann boundary condition on \( l_6 \). Results obtained for the functionals \( J_f \) with Taylor-Hood elements with fixed \( h = 0.04 \) and \( \delta = 0 \). The viscosity is \( \nu = 10^{-4} \) (top), \( \nu = 10^{-3} \) (mid) and \( \nu = 10^{-6} \) (bottom).

<table>
<thead>
<tr>
<th>( \delta )</th>
<th># iter</th>
<th>( \inf J_f )</th>
<th>( e_0^p )</th>
<th>( e_0^f )</th>
<th>( e_{120}^p )</th>
<th>( e_{120}^f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>8</td>
<td>7.054e-24</td>
<td>1.398e-02</td>
<td>8.173e-05</td>
<td>1.539e-04</td>
<td>6.044e-05</td>
</tr>
<tr>
<td>0.1</td>
<td>14</td>
<td>1.005e-25</td>
<td>3.046e-02</td>
<td>1.121e-04</td>
<td>1.233e-04</td>
<td>8.945e-05</td>
</tr>
<tr>
<td>0.05</td>
<td>25</td>
<td>1.388e-23</td>
<td>6.101e-02</td>
<td>1.966e-04</td>
<td>8.335e-05</td>
<td>1.249e-04</td>
</tr>
<tr>
<td>0.02</td>
<td>65</td>
<td>3.579e-23</td>
<td>9.699e-01</td>
<td>3.047e-03</td>
<td>6.782e-05</td>
<td>1.272e-03</td>
</tr>
<tr>
<td>0.01</td>
<td>211</td>
<td>8.323e-21</td>
<td>6.444e+00</td>
<td>1.995e-02</td>
<td>8.936e-05</td>
<td>5.879e-03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \delta )</th>
<th># iter</th>
<th>( \inf J_f )</th>
<th>( e_0^p )</th>
<th>( e_0^f )</th>
<th>( e_{120}^p )</th>
<th>( e_{120}^f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>8</td>
<td>3.724e-25</td>
<td>1.240e-02</td>
<td>5.379e-05</td>
<td>1.271e-04</td>
<td>3.804e-05</td>
</tr>
<tr>
<td>0.1</td>
<td>14</td>
<td>2.542e-25</td>
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</table>

For \( i = 1, 2 \), we consider two unknown control functions \( \lambda_i \in \Lambda_i^D \) and the state problems

\[
- \text{div} T(u_i^{\lambda_i,f}, p_i^{\lambda_i,f}) = f \quad \text{in} \ \Omega_i \\
\text{div} u_i^{\lambda_i,f} = 0 \quad \text{in} \ \Omega_i \\
u_i^{\lambda_i,f} = \lambda_i \quad \text{on} \ \Gamma_i
\]

with suitable homogeneous boundary conditions on \( \partial \Omega_i \setminus \Gamma_i \). If \( \Gamma_N = \emptyset \), we add the constraint \( \int_{\Omega_i} p_i^{\lambda_i,f} = 0 \). The unknown controls on the interface are obtained by solving the minimization problem

\[
\inf_{\Delta \in \Lambda_i^D} \left[ J_f(\Delta) := \frac{1}{2} || u_1^{\lambda_1,f} - u_2^{\lambda_2,f} ||^2_{L^2(\Gamma_1 \cup \Gamma_2)} \right] \]

where, for simplicity of notation, we adopt the same notation as in the discrete case. come possessiamo togliere la ripetizione su "notation"?

This yields an optimal control problem where both the control functions and the observables are of boundary (interface) type.

Thanks to the linearity of the problem, we have \( u_i^{\lambda_i,f} = u_i^{\lambda_i,0} + u_i^{f} \) and \( p_i^{\lambda_i,f} = p_i^{\lambda_i,0} + p_i^{f} \). For simplicity of notation, we will indicate \( u_i^{\lambda_i} = u_i^{\lambda_i,0} \) and \( p_i^{\lambda_i} = p_i^{\lambda_i,0} \) and \( \Delta = (u_1^{\lambda_1}, u_2^{\lambda_2}) \).

Then, we can equivalently express the cost functional as

\[
J_f(\Delta) = \frac{1}{2} || u_1^{\lambda_1,f} - u_2^{\lambda_2,f} ||^2_{L^2(\Gamma_1 \cup \Gamma_2)} + \frac{1}{2} || u_1^{f} - u_2^{f} ||^2_{L^2(\Gamma_1 \cup \Gamma_2)}
\]

In this section we will denote \( ||| \Delta |||_D = || u_i^{\lambda_i} - u_i^{\lambda_2} ||_{L^2(\Gamma_1 \cup \Gamma_2)} \).

**Lemma 5.1** If the boundary conditions imposed on the Stokes problem (1) satisfy Assumption 2.7 then \( ||| \Delta |||_D \) defines a norm on the space \( \Lambda_i^D \).
TABLE XXII: Test case without analytic solution. Neumann boundary condition on \( l_0 \). Results obtained for the functional \( J_{1f} \) with stabilized \( Q_0 - Q_9 \) elements with fixed \( p = 6 \) and \( \delta \to 0 \). The viscosity is \( \nu = 10^{-2} \) (top), \( \nu = 10^{-3} \) (mid), \( \nu = 10^{-6} \) (bottom).

\[
\nu = 10^{-2}
\]

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\[
\nu = 10^{-4}
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\[
\nu = 10^{-6}
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Proof. Since \( \| \lambda \|_{\mathcal{D}} \) is always a semi-norm on \( \Lambda^D \), we only have to prove that, if \( \| \lambda \|_{\mathcal{D}} = 0 \), then \( \lambda = 0 \). Obviously, \( \| \lambda \|_{\mathcal{D}} = 0 \) implies that \( u_1^\lambda = u_2^\lambda \) a.e. on \( \Gamma_1 \cup \Gamma_2 \). We recall that \( u_1^\lambda, u_2^\lambda \) is the solution of (49) with \( f = 0 \). Then, \( (w, q) = (u_1^{\lambda_1}, u_2^{\lambda_2}, u_1^{\lambda_1} - u_2^{\lambda_2}) \) satisfies the problem

\[
\begin{align*}
- \nabla \mathbf{T}(w, q) &= \mathbf{0} \quad \text{in } \Omega_{12} \\
\nabla w &= \mathbf{0} \quad \text{on } \Gamma_1 \cup \Gamma_2 \\
\end{align*}
\]

with suitable homogeneous boundary conditions on \( \partial \Omega_{12} \cap \partial \Omega \). Notice that since \( u_1^{\lambda_1} \) belongs to \( H^1(\Omega_1) \), condition (52), has to be interpreted in the sense of traces of zeroth order of \( H^1 \) functions on \( \Gamma_1 \cup \Gamma_2 \).

Following the same arguments used in the proof of Proposition 2.1, problem (52) is well-posed and its solution is \( w = 0 \) and \( q = \text{const} \). Thus, \( u_1^{\lambda_1} = u_2^{\lambda_2} \) and \( p_0^{\lambda_1} + C_1 = p_2^{\lambda_2} + C_2 \) a.e. in \( \Omega_{12} \) with \( C_1, C_2 \in \mathbb{R} \), and we can define

\[
\mathbf{\tilde{\pi}} = \begin{cases} 
\text{in } \Omega_1 \setminus \Omega_{12} \\
\text{in } \Omega_{12} \\
\end{cases}
\]

and

\[
\mathbf{\pi} = \begin{cases} 
\mathbf{u}_1^{\lambda_1} + C_1 \quad \text{in } \Omega_1 \setminus \Omega_{12} \\
\mathbf{u}_1^{\lambda_1} + \mathbf{u}_2^{\lambda_2} + C_2 \quad \text{in } \Omega_{12} \\
\mathbf{u}_2^{\lambda_2} + C_2 \quad \text{in } \Omega_2 \setminus \Omega_{12} 
\end{cases}
\]

By construction, the pair \((\mathbf{\pi}, \mathbf{\pi})\) satisfies a Stokes problem in \( \Omega \) with null force and homogeneous boundary conditions with \( \Gamma_N \neq \emptyset \). This problem is well-posed and, in particular, \( \mathbf{\pi} = 0 \) a.e. in \( \Omega \). This implies that \( \mathbf{\pi} = 0 \) on \( \Gamma_1 \cup \Gamma_2 \) and, for \( i = 1, 2 \), \( \lambda_i = 0 \) in \( \Lambda_i \).

Although we cannot guarantee that \( \Lambda^D \) is complete with respect to the norm \( \| \lambda \|_{\mathcal{D}} \), we can construct its completion, say \( \hat{\Lambda}^D \), with respect to such norm. In practice, we will always consider a finite dimensional space \( \Lambda^D_h \subset \Lambda^D \subset \hat{\Lambda}^D \) and, at the discrete level, all norms are equivalent. Thus, this does not cause a problem for the application that we have in mind. For the sake of notation, in the following we will still denote the completion of \( \Lambda^D \) by the same symbol.

**Theorem 5.1** Consider the minimization problem

\[
\inf_{\Lambda^D} J_1(\lambda).
\]

If Assumption 2.1 holds, problem (55) has a unique solution satisfying

\[
(\Lambda^D)\left(J_1(\lambda), \mu\right)_{\Lambda^D} = \\
(\mathbf{u}_1^{\lambda_1} - \mathbf{u}_2^{\lambda_2}, \mathbf{u}_1^{\lambda_1} - \mathbf{u}_2^{\lambda_2})_{L^2(\Gamma_1 \cup \Gamma_2)} = 0
\]

for all \( \mu \in \Lambda^D \).

Proof. For any \( \lambda \in \Lambda^D \), let us define

\[
\pi(\lambda, \mu) = \frac{1}{2}(\mathbf{u}_1^{\lambda_1} - \mathbf{u}_2^{\lambda_2}, \mathbf{u}_1^{\mu_1} - \mathbf{u}_2^{\mu_2})_{L^2(\Gamma_1 \cup \Gamma_2)},
\]

\[
L(\mu) = \frac{1}{2}(\mathbf{u}_1^{0_f}, \mathbf{u}_2^{0_f})_{L^2(\Gamma_1 \cup \Gamma_2)}
\]

so that

\[
J_1(\lambda) = \pi(\lambda, \lambda) - 2L(\lambda) + \frac{1}{2}\|\mathbf{u}_1^{0_f} - \mathbf{u}_2^{0_f}\|_{L^2(\Gamma_1 \cup \Gamma_2)}.
\]

The bilinear form \( \pi : \Lambda^D \times \Lambda^D \to \mathbb{R} \) is symmetric by definition and, thanks to Lemma 5.1, is continuous and coercive with respect to the norm \( \| \lambda \|_{\mathcal{D}} \). Moreover, \( L : \Lambda^D \to \mathbb{R} \) is a linear continuous functional. Then, being \((\Lambda^D, \| \cdot \|_{\mathcal{D}})\) an Hilbert space (recall that now \( \Lambda^D \) denotes its completion with respect to the norm \( \| \cdot \|_{\mathcal{D}} \)), applying classical results of calculus of variations (see, e.g., [2, Theorem 1.1]), the existence and uniqueness of the solution is guaranteed.

The Euler-Lagrange equation (56) follows by observing that, for all \( \lambda, \mu \in \Lambda^D \),

\[
(\Lambda^D)\left(J_1(\lambda), \mu\right)_{\Lambda^D} = 2\pi(\lambda, \mu) - 2L(\mu).
\]
Remark 5.1 Notice that, although the definition of the functional $J_i$ involves only the difference between the traces of the velocity on $\Gamma_1 \cup \Gamma_2$, the requirement that $\partial \Omega_{12} \cap \Gamma_N \neq \emptyset$ guarantees that the local pressures $p_1$ and $p_2$ will match in the overlapping region, i.e., $p_1 = p_2$ a.e. in $\Omega_{12}$.

5.1.1. The optimality system for Dirichlet controls

After Theorem 5.1, we assume that Assumption 3.1 is satisfied. More in particular, we consider the case $\partial \Omega_{12} \cap \Gamma_N \neq \emptyset$ and $\Gamma_D \neq \emptyset$ so that the constants $C_1$ and $C_2$ of Lemma 5.1 are both null. In the other cases, we would require that $p_i, q_i \in Q_{i,1}$, and the non-null constants $C_1, C_2$ are those identified in the proof of Proposition 5.1.

The Euler-Lagrange equation (56) becomes:

$$ \left\langle \Lambda, \beta \right\rangle = \int_{\Gamma_1 \cup \Gamma_2} (u^{\alpha^{1, i}} - u^{\alpha^{2, i}}) \cdot (u^{\alpha^{1, i}} - u^{\alpha^{2, i}}) \, d\Gamma = 0 $$

(57)

for all $\mu \in \Lambda^D$.

Solving equation (57) is equivalent to solving the following optimality system: find $\bar{\mu} = (\lambda_1, \lambda_2) \in \Lambda^D$ and, for $i = 1, 2$, $(u_i, p_i) \in V_{i,0} \times Q_{i,1}$, $(w_i, q_i) \in V_{i,0} \times Q_{i,1}$ such that

$$ \begin{aligned}
- \text{div} \, T(u_i, p_i) &= f \quad \text{in} \, \Omega_i \\
\text{div} \, u_i &= 0 \quad \text{in} \, \Omega_i \\
\lambda_i &= 0 \quad \text{on} \, \Gamma_i \\
T(u_i, p_i) \cdot n &= 0 \quad \text{on} \, \Gamma_N
\end{aligned} $$

(58)

$$ \begin{aligned}
- \text{div} \, T(w_i, q_i) &= 0 \quad \text{in} \, \Omega_i \\
\text{div} \, w_i &= 0 \quad \text{in} \, \Omega_i \\
\lambda_i &= 0 \quad \text{on} \, \Gamma_i \\
T(w_i, q_i) \cdot n &= 0 \quad \text{on} \, \Gamma_N
\end{aligned} $$

(59)

and, for all $(\mu_1, \mu_2) \in \Lambda^D$,

$$ \int_{\Gamma_1} ((u_1 - u_2) + w_2) \, d\Gamma + \int_{\Gamma_2} ((u_1 - u_2) + w_1) \, d\Gamma = 0. $$

(60)

Proposition 5.1 The optimality system (58)-(60) has a unique solution whose control component $\bar{\mu} \in \Lambda^D$ is the solution of the Euler-Lagrange equation (57).

Proof. Let $\bar{\mu}$ be the solution of (50). Theorem 5.1 guarantees that such solution exists and is unique. Then, it is also a solution of (58)-(60). Indeed, the solution satisfies (57) which implies that $u^{\alpha^{1, i}} = u^{\alpha^{2, i}}$ on $\Gamma_1 \cup \Gamma_2$. As a consequence the solutions $(w_i, q_i)$ of (59) are identically null and (60) is satisfied.

We prove now that this solution is unique. Consider first the case $f = 0$. We define the operator $\chi : \Lambda^D \to (\Lambda^D)'$,

$$ \left\langle \Lambda^D, (\chi(\Lambda), \mu) \right\rangle = \int_{\Gamma_1} (u^{\alpha^{1, i}} - u^{\alpha^{2, i}}) \cdot (u^{\alpha^{1, i}} - u^{\alpha^{2, i}}) \, d\Gamma + \int_{\Gamma_2} ((u_1 - u_2) + w^{\alpha^{1}}) \, d\Gamma $$

(61)

where, for $i = 1, 2$, $u^{\alpha_{i}}$, $w^{\alpha_{i}}$ are solutions of (58) and (59), respectively, with $f = 0$. The operator $\chi$ is linear and continuous, and $\ker(\chi) = \{0\}$. Indeed, thanks to (59), $w^{\alpha_{i}} \in [H^{1/2}(\Gamma_i)]^2$ and, if $\lambda \in \ker(\chi)$, due to (61), $w^{\alpha_{i}} = -(u^{\alpha_{i}} - u^{\alpha_{i}})$ on $\Gamma_1$ and $w^{\alpha_{i}} = (u^{\alpha_{i}} - u^{\alpha_{i}})$ on $\Gamma_2$.

Thus, for $i = 1, 2, j = 3 - i$, $w^{\alpha_{i}}$ satisfies the system

$$ \begin{aligned}
- \text{div} \, T(w^{\alpha_{i}}, q^{\alpha_{i}}) &= 0 \quad \text{in} \, \Omega_i \\
\text{div} \, w^{\alpha_{i}} &= 0 \quad \text{in} \, \Omega_i \\
w^{\alpha_{i}} &= -w^{\alpha_{i}} \quad \text{on} \, \Gamma_i \\
w^{\alpha_{i}} &= 0 \quad \text{on} \, \Gamma_D \\
T(w^{\alpha_{i}}, q^{\alpha_{i}}) \cdot n &= 0 \quad \text{on} \, \Gamma_N
\end{aligned} $$

We define $\bar{w} = w^{\alpha_{1}}|_{\Omega_{12}} + w^{\alpha_{2}}|_{\Omega_{12}}$ and $\bar{q} = q^{\alpha_{1}}|_{\Omega_{12}} + q^{\alpha_{2}}|_{\Omega_{12}}$ in $\Omega_{12}$. By construction $(\bar{w}, \bar{q}) \in V_{1,2} \times L^2(\Omega_{12})$ and they satisfy the problem

$$ \begin{aligned}
- \text{div} \, T(\bar{w}, \bar{q}) &= 0 \quad \text{in} \, \Omega_{12} \\
\text{div} \, \bar{w} &= 0 \quad \text{in} \, \Omega_{12} \\
\bar{w} &= 0 \quad \text{on} \, \Gamma_1 \cup \Gamma_2 \\
\bar{w} &= 0 \quad \text{on} \, \Gamma_D \cap \partial \Omega_{12} \\
T(\bar{w}, \bar{q}) \cdot n &= 0 \quad \text{on} \, \Gamma_N \cap \partial \Omega_{12}
\end{aligned} $$

whose solution is identically null. Thus, $w^{\alpha_{1}} = -w^{\alpha_{2}}$ and $q^{\alpha_{1}} = -q^{\alpha_{2}}$ in $\Omega_{12}$ and we can define

$$ \begin{aligned}
w &= \begin{cases} w^{\alpha_{1}} & \text{in} \, \Omega_{12} \\
w^{\alpha_{2}} & \text{in} \, \Omega_{12} \setminus \Omega_{12} \\
w^{\alpha_{2}} & \text{in} \, \Omega_{12} \setminus \Omega_{12}
\end{cases}
\end{aligned} $$

and

$$ \begin{aligned}
q &= \begin{cases} q^{\alpha_{1}} & \text{in} \, \Omega_{12} \\
q^{\alpha_{2}} & \text{in} \, \Omega_{12} \setminus \Omega_{12} \\
q^{\alpha_{2}} & \text{in} \, \Omega_{12} \setminus \Omega_{12}
\end{cases}
\end{aligned} $$

which satisfy the Stokes problem

$$ \begin{aligned}
- \text{div} \, T(w, q) &= 0 \quad \text{in} \, \Omega \\
\text{div} \, w &= 0 \quad \text{in} \, \Omega \\
w &= 0 \quad \text{on} \, \Gamma_D \\
T(w, q) \cdot n &= 0 \quad \text{on} \, \Gamma_N
\end{aligned} $$

whose unique solution is $w = 0$ and $q = 0$. Thus, we can conclude that $w^{\alpha_{1}} = 0$ in $\Omega_i$ ($i = 1, 2$) and $u^{\alpha_{i}} = u^{\alpha_{2}}$ on $\Gamma_1 \cup \Gamma_2$. 


Applying a similar argument to the state equations \(\ref{eq:state_equations} \) with \( f = 0 \) and defining \( \tilde{w} = w^\Delta_{1|\Omega_{12}} - w^\Delta_{2|\Omega_{12}} \) and \( \tilde{q} = q^\Delta_{1|\Omega_{12}} - q^\Delta_{2|\Omega_{12}} \) in \( \Omega_{12} \), we can prove that both these functions are null and can conclude that \( \lambda = 0, i = 1, 2 \).

If \( f \neq 0 \), for \( i = 1, 2 \) and \( j = 3 - i \), let \( w_f^i \), \( q_f^i \) be the solution of the problem

\[
\begin{align*}
    -\nabla T(w_f^i, q_f^i) &= 0 \quad \text{in } \Omega_i \\
    \nabla w_f^i &= 0 \quad \text{in } \Omega_i \\
    w_f^i &= u_{0,f}^i - u_{0,r}^i \quad \text{on } \Gamma_i \\
    w_f^i &= 0 \quad \text{on } \Gamma_D \\
    T(w_f^i, q_f^i) \cdot n &= \lambda_i \quad \text{on } \Gamma_i
\end{align*}
\]

\( u_{0,f}^i \) being the solutions of \(\ref{eq:state_equations} \) with \( \lambda_i = 0 \). Then, we can write \(\ref{eq:optimality_system} \) as

\[
\langle A_f \rangle (\lambda, \mu)_{\Lambda^D} = -\langle A_f \rangle (\mu, \lambda)_{\Lambda^D}, \quad \forall \mu \in \Lambda^D,
\]

where

\[
\begin{align*}
    A_f : \Lambda^D &\rightarrow (\Lambda^D)', \\
    (A_f)'(\lambda, \mu)_{\Lambda^D} &= \int_{\Gamma_1} ((u_{1,0}^f - u_{2,0}^f) + w_f^1) \mu_1 \, d\Gamma \\
    &\quad + \int_{\Gamma_2} (-u_{1,0}^f - w_f^2) \mu_2 \, d\Gamma.
\end{align*}
\]

The theorem follows from the same arguments used before. \(\blacklozenge\)

Since the space \( \Lambda^D \) of discrete Dirichlet controls is a subset of \( \Lambda^D \), Lemma \(\ref{lem:optimality_system} \) Theorem \(\ref{thm:optimality_system} \) and Proposition \(\ref{prop:optimality_system} \) hold in the discrete case too and then, we can conclude that the discrete minimization problem \(\ref{eq:discrete_minimization} \) - \(\ref{eq:discrete_minimization} \), or equivalently the associated optimality system \(\ref{eq:optimality_system} \) - \(\ref{eq:optimality_system} \) has a unique solution.

The minimum of the cost functional \( J_\epsilon \) is zero thanks to Proposition \(\ref{prop:optimality_system} \). The real value \(\inf_{\lambda_i \in \Lambda^D} J_\epsilon (\lambda_i) \) attained at convergence, and reported in the second column of the Tables \(\ref{table:comparison} \) and \(\ref{table:comparison} \) is about \( \epsilon^2 \), \( \epsilon = 10^{-9} \) being the tolerance in the stopping criterium of Bi-CGSTab iterations. We notice that by reducing the tolerance \( \epsilon \), \(\inf_{\lambda_i \in \Lambda^D} J_\epsilon (\lambda_i) \) reduces too. The errors between the discrete states \( (u_i, p_i) \) and the exact ones \( (u_i, p_i) \) vanish for \( h \to 0 \) and increasing \( p \), according to the theoretical convergence rate of \( hp \)-finite element approximation.

5.2. Analysis of the optimal control problem with Neumann controls

Tutta questa sezione è nuova, anche se è in nero

For \( i = 1, 2 \), let

\[
\Lambda_i^N = [H^{-1/2}(\Gamma_i)]^d
\]

denote the spaces of admissible Neumann controls and we set

\[
\Lambda^N = \Lambda_i^N \times \Lambda_j^N.
\]

For \( i = 1, 2 \), we consider two unknown control functions \( \lambda_i \in \Lambda_i^N \) and the state problems

\[
\begin{align*}
    -\nabla T(u_{1,\lambda}^i, p_{1,\lambda}^i) &= f \quad \text{in } \Omega_i \\
    \nabla u_{1,\lambda}^i &= 0 \quad \text{in } \Omega_i \\
    T(u_{1,\lambda}^i, p_{1,\lambda}^i) \cdot n &= \lambda_i \quad \text{on } \Gamma_i
\end{align*}
\]

with suitable homogeneous boundary conditions on \( \partial \Omega_i \setminus \Gamma_i \). The unknown controls on the interface are obtained by solving the minimization problem

\[
\inf_{\Delta = (\lambda_1, \lambda_2)} \left[ \frac{1}{2} \| T(u_{1,\lambda}^i, p_{1,\lambda}^i) \cdot n \|_{H^{-1/2}(\Gamma_i)}^2 \right].
\]

Denoting by \( -\Delta_{\Gamma_i} \) the Laplace Beltrami operator on \( \Gamma_i \), for any \( \psi, \phi \in H^{-1/2}(\Gamma_i) \) we define the following inner product (see, e.g., \([2]\)):

\[
\langle \psi, \phi \rangle_{H^{-1/2}(\Gamma_i)} = \int_{\Gamma_i} \langle -\Delta_{\Gamma_i} \rangle^{-1/2} \psi \Delta_{\Gamma_i}^{-1/2} \phi \, d\Gamma_i
\]

and the related norm \( \| \psi \|_{H^{-1/2}(\Gamma_i)} = \langle \psi, \psi \rangle_{H^{-1/2}(\Gamma_i)}^{1/2} \).

The fractional Laplace-Beltrami operator \( -\Delta_{\Gamma_i}^{-1/2} \) can be defined through a Neumann to Dirichlet map defined from \( H^{-1/2}(\Gamma_i) \) to \( H^{1/2}(\Gamma_i) \) (see, e.g., \([1]\)). Precisely, for any \( \phi \in H^{-1/2}(\Gamma_i) \) we solve the problem

\[
\begin{align*}
    -\Delta u + u &= 0 \quad \text{in } \Omega_i \\
    \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega_i \setminus \Gamma_i \\
    \frac{\partial u}{\partial n} &= \phi \quad \text{on } \Gamma_i
\end{align*}
\]

and we set \( -\Delta_{\Gamma_i}^{-1/2} \phi = u|_{\Gamma_i} \).

From now on, let \( (-, -)_{-1/2} \) and \( \| \cdot \|_{-1/2} \) replace \( (-, -)_{H^{-1/2}(\Omega_i \cup \Gamma_i)} \) and \( \| \cdot \|_{H^{-1/2}(\Omega_i \cup \Gamma_i)} \), respectively. \(\ref{eq:optimality_system} \) is an optimal control problem where both the control functions and the observations are of boundary (interface) type.

As for Dirichlet case, thanks to the linearity of the problem, we can equivalently express the cost functional as

\[
\begin{align*}
    \tilde{J}_f(\lambda) &= \frac{1}{2} \| T(u_{1,\lambda}^1, p_{1,\lambda}^1) \cdot n - T(u_{2,\lambda}^2, p_{2,\lambda}^2) \cdot n \|_{-1/2} \\
    &\quad + \| T(u_{1,\lambda}^1, p_{1,\lambda}^1) \cdot n - T(u_{2,\lambda}^2, p_{2,\lambda}^2) \cdot n \|_{-1/2} \\
    &\quad + \frac{1}{2} \| T(u_{1,\lambda}^1, p_{1,\lambda}^1) \cdot n - T(u_{2,\lambda}^2, p_{2,\lambda}^2) \cdot n \|_{-1/2} \\
    &\quad + \frac{1}{2} \| T(u_{1,\lambda}^1, p_{1,\lambda}^1) \cdot n - T(u_{2,\lambda}^2, p_{2,\lambda}^2) \cdot n \|_{-1/2}
\end{align*}
\]

Let us denote

\[
\| \lambda \|_{N} = \| T(u_{1,\lambda}^1, p_{1,\lambda}^1) \cdot n - T(u_{2,\lambda}^2, p_{2,\lambda}^2) \cdot n \|_{-1/2}.
\]
Lemma 5.2 If $\partial \Omega \cup \Gamma_D \neq \emptyset$, then $|||\Delta|||_N$ defines a norm on the space $\Lambda^N$.

Proof. We proceed as for Dirichlet controls: $|||\Delta|||_N$ is always a semi-norm on $\Lambda^N$, we only have to prove that, if $|||\Delta|||_N = 0$, then $\Delta = 0$. Obviously, $|||\Delta|||_N = 0$ implies that $T(u^{x_1}_h, p^{y}_h \cdot n) = T(u^{x_2}_h, p^{y}_h \cdot n) = 0$ a.e. on $\Gamma_1 \cup \Gamma_2$. In view of Proposition 5.2, starting from $(u^{x}_h, p^{y}_h)$ we define the pair $(\overline{u}, \overline{p})$ as in (53), (54), that satisfies a Stokes problem in $\Omega$ with null force and homogeneous boundary conditions. This problem is well-posed and, in particular, $\overline{u} = 0$ and $\overline{p} = 0$ a.e. in $\Omega$. This implies that $T(\overline{u}, \overline{p}) \cdot n = 0$ on $\Gamma_1 \cup \Gamma_2$ and, for $i = 1, 2$, $\lambda_i = 0$ in $\Lambda^N$.

We cannot guarantee that $\Lambda^N$ is complete with respect to the norm $|||\Delta|||_N$, but we can construct its completion, say $\tilde{\Lambda}^N$, with respect to such norm. For the sake of notation, in the following we will still denote the completion of $\Lambda^N$ by the same symbol.

Theorem 5.2 Consider the minimization problem

$$\inf_{\Delta \in \Lambda^N} \tilde{J}_f(\Delta),$$

If $\partial \Omega \cup \Gamma_D \neq \emptyset$, problem (69) has a unique solution satisfying

$$(\Lambda^N) / \tilde{J}_f(\Delta), \mu \Lambda^N = (T(u^{x_1}_h, p^{y}_h \cdot n) - T(u^{x_2}_h, p^{y}_h \cdot n), \mu)$$

(70)

$$T(u^{x}_h, \mu), \mu) = (T(u^{x_2}_h, \mu), \mu) - (T(u^{x}_h, \mu), \mu) - 1/2 = 0$$

for all $\mu \in \Lambda^N$.

Proof. The proof follows the same guidelines of the proof of Theorem 5.1.

In view of (66), the Euler-Lagrange equation (70) becomes:

$$\sum_{i=1}^{2} \int_{\Gamma_i} (-\Delta_{\overline{\Gamma}},) - 1/2 (T(u^{x_1}_h, p^{y}_h \cdot n) - T(u^{x_2}_h, p^{y}_h \cdot n) \cdot n + T(u^{x_2}_h, p^{y}_h \cdot n) \cdot n) \cdot n) = 0$$

(71)

for all $\mu \in \Lambda^N$ and $j = 3 - i$.

Solving (equation 71) is equivalent to solving the following optimality system: find $\lambda = (\lambda_1, \lambda_2) \in \Lambda^N$ and, for $i = 1, 2$, $(u_i, p_i) \in V_{i,0} \times Q_{i,0}$, $(w_i, q_i) \in V_{i,0} \times Q_{i,0}$, such that

$$-\text{div} \ T(u_i, p_i) = 0 \quad \text{in } \Omega_i$$

$$\text{div} \ u_i = 0 \quad \text{in } \Omega_i$$

$$T(u_i, p_i) \cdot n = \lambda_i \quad \text{on } \Gamma_i$$

$$T(u_i, p_i) \cdot n = 0 \quad \text{on } \Gamma^i_N$$

(72)

$$-\text{div} \ T(w_i, q_i) = 0 \quad \text{in } \Omega_i$$

$$\text{div} \ w_i = 0 \quad \text{in } \Omega_i$$

$$T(w_i, q_i) \cdot n = (-1)^{i+1} (T(u_i, p_i) \cdot n$$

$$- T(u_j, p_j) \cdot n) \quad \text{on } \Gamma_i$$

$$T(w_i, q_i) \cdot n = 0 \quad \text{on } \Gamma^i_N$$

(73)

and, for all $(\mu_1, \mu_2) \in \Lambda^N$,

$$\sum_{i=1}^{2} \int_{\Gamma_i} (-\Delta_{\overline{\Gamma}},) - 1/2 (T(u_i, p_i) \cdot n - T(u_j, p_j) \cdot n$$

$$+ T(w_j, q_j) \cdot n) \mu_i \cdot n = 0$$

(74)

for $j = 3 - i$.

Proposition 5.2 The optimality system (72), (74) has a unique solution whose control component $\lambda \in \Lambda^N$ is the solution of the Euler-Lagrange equation (71).

Proof. Let $\Delta$ be the solution of (69). Theorem 5.2 guarantees that such solution exists and is unique. Then, it is also a solution of (72), (74). Indeed, the solution satisfies (71) which implies that $T(u^{x}_h, p^{y}_h \cdot n) = T(u^{x}_h, p^{y}_h \cdot n)$ on $\Gamma_1 \cup \Gamma_2$. As a consequence the solutions $(w_i, q_i)$ of (73) are identically null and (74) is satisfied.

To prove that this solution is unique, we proceed as in the proof of Proposition 5.1 by exploiting linearity, continuity and coercivity of the Laplace-Beltrami operator (see (62)).

In view of Proposition 2.2 the infimum of $\tilde{J}_f$ is zero.

The cost function $J_f$ differs from $J_f$ defined in (24) for the choice of the norm. As a matter of fact, at the continuous level we cannot guarantee that the fluxes $T(u^{x}_h, p^{y}_h \cdot n) \cdot n$ are $L^2$ functions, but their natural space is $[H^{1/2}(\Gamma_1 \cup \Gamma_2)]^d$, while the discrete fluxes are more regular and belong to $[L^2(\Gamma_1 \cup \Gamma_2)]^d$, as we shown in Section 3.3.3.

Since the space $\Lambda^N$ of discrete Neumann controls is a subset of $\Lambda^N$, we can conclude that also the minimization problem

$$\inf_{\Delta \in \Lambda^N} \tilde{J}_f(\Delta)$$

(75)

has a unique solution (thanks to Lemma 5.2 and Theorem 5.2) and it can be computed by solving the optimality system (72), (74) (by Proposition 5.2).

Following the same guidelines of Lemmas 5.2 and Theorem 5.2, we can prove that also the minimization problem (24) has a unique solution.

At the discrete level the solutions computed by solving (24) and (75) could not coincide, nevertheless, results of Tables VI and VII show that

$$\inf_{\lambda \in \Lambda^N} J_f(\lambda_h) \to 0 \text{ when } h \to 0$$

(76)
for Taylor-Wood discretization, while \( \inf_{\lambda_h \in \Lambda^h_N} J_f(\lambda_h) \simeq \epsilon^2 \) for \( \mathbb{Q}_p - \mathbb{Q}_p \) discretizations, where \( \epsilon = 10^{-9} \) is the tolerance used in the stopping test of Bi-CGStab iterations. Moreover, as for Dirichlet controls, the errors between the discrete states \((u_i,h, p_i,h)\) and the exact ones \((u_i, p_i)\) vanish for \( h \to 0 \) and increasing \( p \), according with the theoretical convergence rate of \( hp \)-finite element approximation.

For what concerns the minimization problems \( (30) \) and \( (31) \) with mixed controls, we can apply the analysis developed for both Dirichlet and Neumann controls and draw the same conclusions given above for the cost functional \( J_f \).

6. CONCLUSIONS

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